

Weak Limits of Riemannian Metrics in Surfaces with integral Curvature Bound

Xiuxiong Chen

Department of Mathematics

Stanford University

Palo Alto, CA 94305

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1 Introduction

1.1 Introduction to the problem. We study the limit of a sequence of Riemannian metrics on a surface under some suitable conditions. Let Ω be any open domain and let $\mathcal{G}(\Omega)$ be the set of all smooth Riemannian metrics on Ω . Any two metrics g_1, g_2 in $\mathcal{G}(\Omega)$ are called pointwise conformally equivalent if they are related under multiplication by a smooth, positive function on Ω . This relation is denoted by $g_1 \propto g_2$. Define the curvature energy function and area function for a metric g in Ω as follows:

$$K(g, \Omega) = \int_{\Omega} K_g^2 d g, \quad A(g, \Omega) = \int_{\Omega} d g,$$

where K_g is the scalar curvature of g , and $d g$ is the area (volume) element. For a given metric g_0 in Ω , define a function space $\mathcal{S}(g_0, C_1, C_2, \Omega)$ to be the completion of the following set under any reasonable topology:

$$\mathcal{S}(g_0, C_1, C_2, \Omega) = \{g \in \mathcal{G}(\Omega) | g \propto g_0, A(g, \Omega) = C_1, K(g, \Omega) \leq C_2\},$$

here C_1 and C_2 are generic constants.

We are mainly concerned with the two following questions: (a) Given a sequence of metrics $\{g_k, k \in \mathbf{N}\}$ in $\mathcal{S}(g_0, C_1, C_2, \Omega)$, what is the set of its cluster points? (b) When can one conclude that there must exist at least one limit and, if so, what are its geometric properties? We have constructed an example of a sequence of metrics with no subsequence that converges in the elementary sense. Therefore, one must devise a geometrically reasonable topology in the function space of metrics; in particular, the area functional is continuous and the energy functional is lower-semi-continuous.

Our main result is Theorem A at section 4. It could be summarized as the following: As is shown in Figure 1, there is a subsequence of $\{g_n\}$ which locally weakly converges in $H^{2,2}$ (functions up to second derivative are in L^2) to a Riemannian metric f_0 . However, this weak convergence is not on all of the surface Ω , but on Ω with a number of points $\{p_i\}$ deleted. There is a positive amount of energy and area concentrations at each point p_i . At each point p_i , we use a rescaling argument to construct a sequence of Riemannian metrics in S^2 with a small disk deleted (the size of the disk approaches zero when the sequence takes a limit). This renormalized sequence of metrics then (have a subsequence) converges to a metric f_i in S^2 with a finite number of points deleted. We then call this metric a “bubble metric.” Iterating this process at each new bubble point of f_i , and so on. The final “limit” of the subsequence (passing to the diagonal subsequence) is a disjoint union of these bubble metrics, which are defined in different surfaces. Each metric in the “limit” has a special property that if it vanishes at one point in its domain, it then vanishes everywhere in its domain. While a bubble metric might be a metric in 2-sphere with constant curvature, generically it is a metric defined on a punctured sphere and it has a singular angle at each punctured point.

1.2 Extremal Kähler metrics. The proposed problem is motivated from the study of the existence problem of extremal Kähler metric in a Kähler manifold M . An extremal Kähler metric is a critical point for the energy functional:

$$E(g) = \int_M K_g^2 dg$$

on the space of Kähler metrics in a fixed Kähler class. The Euler equation for the critical metric is (assume $\partial M = \emptyset$):

$$K_{g,\alpha\beta} = 0, \quad \forall 1 \leq \alpha, \beta \leq n,$$

where n is the complex dimension of the manifold.

This problem first appeared in [2]. E. Calabi proposed to use the heat flow method to solve this problem when the manifold admits no holomorphic vector field. The heat flow he suggested is:

$$\frac{\partial g_{\alpha\bar{\beta}}}{\partial t} = K_{g,\alpha\bar{\beta}}, \quad \forall 1 \leq \alpha, \beta \leq n.$$

This flow indeed decreases the energy function E along its trajectory. To obtain the long term existence and the convergence as $t \rightarrow \infty$, one needs to understand the following question: what is the weak topology of the set of metrics in a fixed Kähler class with bounded energy? An initial approach to this question would be to study it in the case of complex dimension one, reducing the problem to the one just described. Observe that any Riemannian metric on a surface is also a Kähler metric; any two metrics g_1, g_2 are in the same Kähler class if and only if $g_1 \propto g_2$

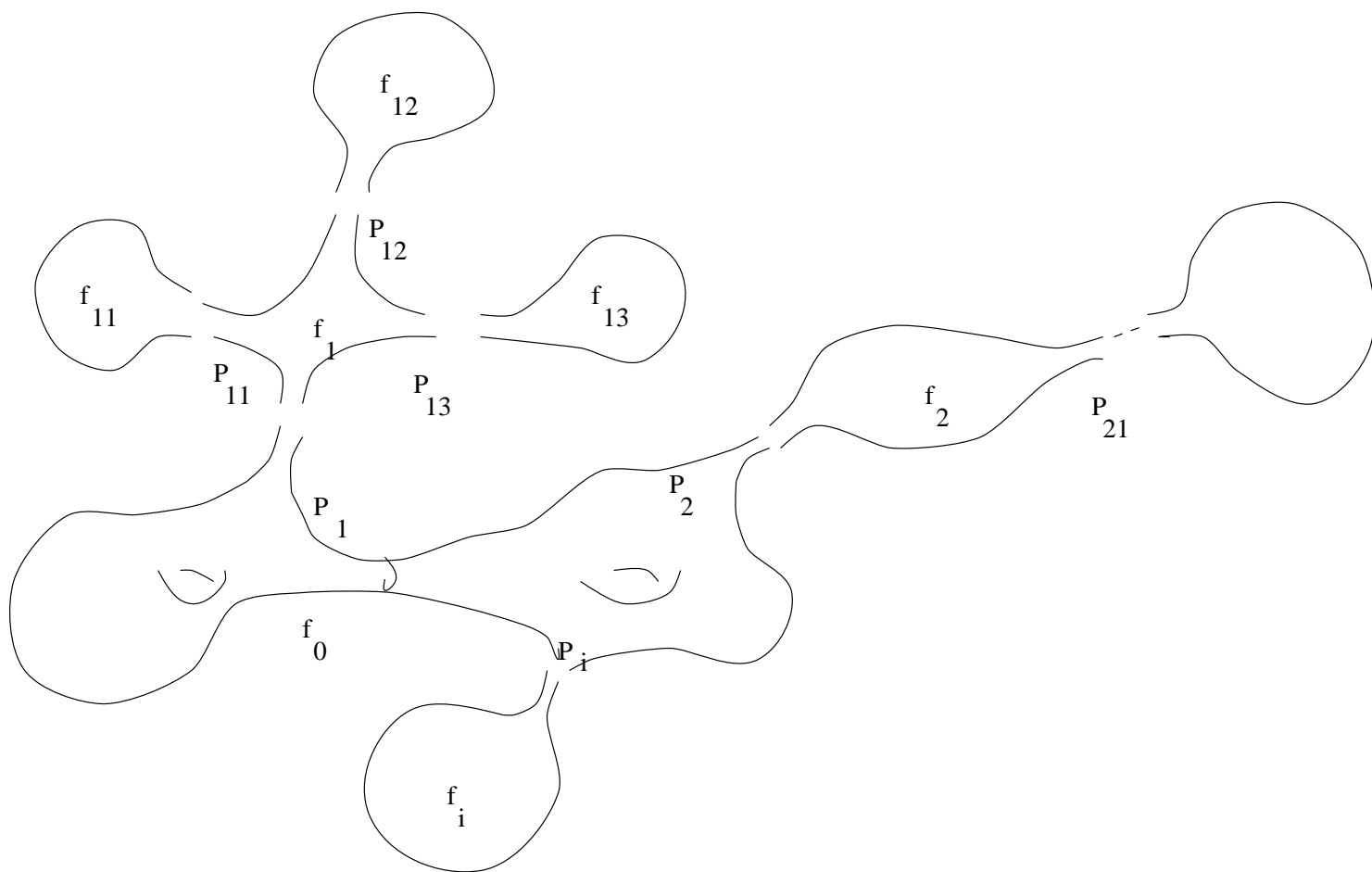


Figure 1: Bubbles on Bubbles

and $\int_{\Omega} d g_1 = \int_{\Omega} d g_2$ if $\partial\Omega = \emptyset$.

1.3 Uniformization theorem and Dirichlet Problem. The selection of L^2 norm (rather than any L^p norm with $p > 1$) of the scalar curvature as energy function is not essential as far as the weak topology of the function space is concerned. It is significant, however, if we consider the corresponding variational problem. The Euler equation of the energy functional is:

$$\Delta_g K_g + K_g^2 = C \text{ (generic constant)}. \quad (1)$$

This Equation is called the extremal equation. Any metric satisfies equation (1) is called an extremal metric, even if it is only a stationary point of the energy functional.

Let Ω be any domain with smooth boundary; let g_0 be a smooth metric in Ω which could be extended smoothly to a slight larger domain. We want to ask if there always exists a metric g , in a pointwise conformal class of g_0 , which solves equation (1) and satisfies the Dirichlet boundary condition:

$$g|_{\partial\Omega} = g_0|_{\partial\Omega}, \quad \frac{\partial g}{\partial n}|_{\partial\Omega} = \frac{\partial g_0}{\partial n}|_{\partial\Omega}. \quad (2)$$

Conjecture 1. *There always exists a solution to equation (1) with Dirichlet boundary condition (2), while solution metric is pointwise conformal to the initial metric g_0 .*

The Euler equation (1) has an equivalent complex version:

$$\frac{\partial}{\partial \bar{z}} K_{g,zz} = 0, \quad K_{g,zz} = \frac{\partial^2 K_g}{\partial z^2} - 2 \cdot \frac{\partial K_g}{\partial z} \cdot \frac{\partial \varphi}{\partial z}, \quad (3)$$

where $g = e^{2\varphi} |dz|^2$ locally.

The Euler equation has two important special cases: the first special case is the following

$$K_{g,zz} = 0, \quad (4)$$

while the second special case is the following

$$K_g \equiv C, \quad \text{or} \quad -\Delta \varphi = C \cdot e^{2\varphi}. \quad (5)$$

Any metric solves the equation (4) has a special property that the Hessian of its curvature is proportional to the metric tensor. Therefore, we may denote these metrics as HCMU metrics (“Hessian of Curvature of Metric is Umbilical”). If the Conjecture 1 were established, it would be desirable to understand the obstructions for the existence of any HCMU metric and obstructions to the existence of any constant curvature metric in a domain with appropriate Dirichlet boundary

condition (2).

In the special case when $\partial\Omega = \emptyset$, any extremal metric also has a constant curvature. Recall that the classical uniformization theorem in a surface with no boundary asserts that any Riemannian metric is pointwise conformal to a metric with constant curvature. Therefore, the Conjecture 1 (if proved), would generalize the classical uniformization theorem in a surface with no boundary to any domain with smooth boundaries.

Consider another special case where the boundaries are a set of isolated points. To replace the Dirichlet boundary conditions, one requires all of the metrics have a prescribed conical angle at each boundary point. Such a surface is called “a surface with conical singularities” (see [9] for definition).

Open Problem 1. *Is any Riemannian metric on a singular surface pointwise conformal to an extremal metric with the same angle at each singular point.*

In this special case, there have been plenty of attempts (mostly by analysts) to generalize the classical uniformization theorem to surfaces with conical singularities. Most work has concentrated on finding a constant curvature metric in a pointwise conformal class. However, we believe our approach may be more fruitful, since not all surfaces with conical singularities support a constant scalar curvature metric. Our program involves two related but independent problems. The first problem is to use direct variational method to give a positive answer to the above problem. For this purpose, we need to study the weak compactness of the function space of Riemannian metrics with finite energy and area (which is the subject of this study). The second problem is to study the obstructions of existence of any HCMU metric and constant curvature metric in such surfaces. This second problem is discussed in [14], where we give a necessary condition for these surfaces to admit any HCMU metric with non-constant curvature.

1.4 Bubbling phenomenon. An important feature of Theorem A is “bubbling phenomenon.” The bubbling phenomenon was first observed by Sacks-Uhlenbeck [13] in 1979, when they studied the existence theorem for harmonic maps between two spheres. Since then, it has been studied and recognized in a wide variety of geometric differential equations (see [6] for further references). The solution spaces to these equations are non-compact in any reasonable topology. The key observation was that the non-compactness is associated with the concentration of the energy density at isolated points and that, by using the conformal invariance of the equations, one could renormalized the solutions around these points to obtain other solutions. This re-normalization process is commonly referred to as “bubbling.”

Our “bubbling” procedure appears to be very similar to the re-normalization process employed first by Sacks-Uhlenbeck in 1979. However, there are some significant differences. First, the function space is not a solution space of any elliptic equation. Second, in most geometric problems where the bubbling phenomenon occurs, the energy function involves only the first derivatives of the “function” in the solution space. However, the energy functional here involves the second derivatives. To make the matter worse, it involves only the Laplacian of the conformal parameter function, which exerts a very weak control on the size of the metric. These differences dictate a new approach other than the standard one to solve the problem. For instance, in most of these problems where bubbling phenomenon occurs, one usually obtains a weak convergence result without too much difficulty. The hard part is to show that the bubble points are isolated. However, we have to do it exactly in the opposite order here. The definition of a “bubble point” then becomes rather tricky, because there is no weak convergent subsequence to work with. To overcome this difficulty, we introduce the notion of “pseudo bubble point,” where a subsequence of metrics has a positive amount of energy and area concentration. Unfortunately, the set of pseudo bubble points could be a dense set in the domain.

1.5 Thick-thin Decomposition. In a higher dimensional compact manifold, the Cheeger-Gromov theorem [3] states that any sequence of metrics in a compact manifold has a convergent subsequence, provided that the sectional curvature is uniformly bounded, the volume is bounded from below, and the diameter is bounded from above. Similar results to [3] were obtained in [7],[10] and [11] as well. The following corollary of the theorem A could be regarded as a 2 dimensional version of the Cheeger-Gromov thick-thin decomposition theorem, under a weaker integral condition on the curvature tensors.

Corollary B. *For any locally weakly convergent sequence of surfaces $\{(\Omega, g_n), n \in \mathbf{N}\}$ where $g_n \in \mathcal{S}(g_0, C_1, C_2, \Omega)$, and for any number $\epsilon > 0$, there exist two integers N_{thick} and N_{thin} which depend only on ϵ and the total energy $\sqrt{C_1 \cdot C_2}$ of this sequence (independent of n). There exists a decomposition of (Ω, g_n) into N_{thick} of thick components $\{(\Omega_\alpha, g_n|_{\Omega_\alpha})\}$ (indexed by I_{thick}) and N_{thin} of thin components $\{(\Omega_\beta, g_n|_{\Omega_\beta})\}$ (indexed by I_{thin}), such that (see Figure 5 on p. 40): 1) $\Omega = \bigcup_{\alpha \in I_{thick}} \Omega_\alpha \bigcup_{\beta \in I_{thin}} \Omega_\beta$. 2) For any fixed $\alpha \in I_{thick}$, except one, $(\Omega_\alpha, g_n|_{\Omega_\alpha})$ locally weakly converges to a metric in S^2 with a finite number of small disks deleted; the other thick component locally weakly converges to a metric in Ω with a finite number of disks deleted. Moreover, the size of each deleted disk could be made as small as needed. 3) Each thick component is self-connected; however, they are mutually disconnected if all of the thin components are removed from the surface. 4) Each thin component is topologically $S^1 \times (a, b)$ and the length of any concentric circle $S^1 \times \{x\} (x \in (a, b))$ is strictly less than ϵ .*

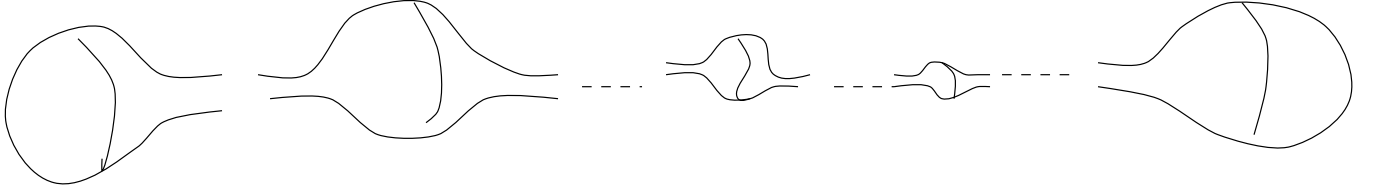


Figure 2: Rotationally symmetric Bubbles

Remark 1. *The terms “thin” and “thick” used here, are slightly different from what are originally used in [3]. For instance, the metrics in a thin part in above corollary do not necessary have a lower bound on the scalar curvature.*

We initially hoped that both numbers N_{thin} and N_{thick} would be independent of ϵ . However, we have constructed a sequence of rotationally symmetric metrics in $\mathcal{S}(g_0, C_1, C_2, S^2)$ such that this sequence yields as many thick components as needed when $\epsilon \rightarrow 0$, without incurring a blowing up of the energy functional. We first constructs a sequence of metrics in a sequence of disks where the boundary of each disk is a smooth closed geodesic; the length of the boundary geodesic tends to 0, while both the energy functional and area are kept uniformly bounded from above (see example 2 in p. 35 for details). We then construct a sequence of metrics in a sequence of cylinder where both boundary circles are geodesics; the length of the boundary geodesics tends to zero, while the energy functional and area functional could make to be arbitrarily small. Using these metrics as building block, we could construct a sequence of metrics with bounded energy and area as in Figure 2, where the limit of metrics splits into as many parts as desired. Henceforth, Corollary **B** in its present form is the best one we could expect.

Motivated by the work of [3], C. Barvard and P. Pansu [5] studied the divergence problem of a sequence of metrics in any surface with pointwise curvature bounded, allowing the conformal structure to be varied. They have constructed some examples which show that the compactness fails if the conformal structure is not bounded. As a matter of fact, the weak compactness still fails even if the conformal structure is fixed. Following the work of C. Barvard and P. Pansu [5], M. Trojanov [12] first considered a sequence of Riemannian metrics in a surface with a L^p ($\forall p > 1$) norm of curvature (with respect to a fixed background metric) uniformly bounded from above. He then showed such a sequence of metrics either has a convergent subsequence or has at least one singular point. However, the “bubbles on bubbles” phenomenon is not observed in [12]

1.6 Analytical approach. In a local coordinate system, any Riemannian metric g could be expressed in terms of its conformal parameter function φ :

$$g = e^{2\varphi}(dx^2 + dy^2).$$

Therefore, g can then be regarded as a solution of scalar curvature equation:

$$-\Delta \varphi = K \cdot e^{2\varphi}. \quad (6)$$

H. Brezis and F. Merle [1] had studied the weak compactness of the solution space of this equation. They consider a sequence of pointwise conformal metrics in an open disk. It is assumed that the $L^p(\forall p > 1)$ norm of curvature is uniformly bounded from above and the curvature function is non-negative. They [1] observed only the first level of bubbles, but not bubbles on bubbles phenomenon.

In both problems, difficulties arise because the right side of equation (6) is only in L^1 . Interested readers are encouraged to compare the main theorems of [1] with the Theorem 1 and 3 in Section 3, where the problem is discussed from an analytic perspective. There are some striking similarities which underscore the connections between these two problems. However, there are also some fundamental differences between these two problems. It is assumed in [1] that either the scalar curvature function is non-negative, or the area element is in $L^{p'}(\frac{1}{p} + \frac{1}{p'} = 1)$. The compactness fails in our problem precisely because that the scalar curvature function changes sign and the area element is only in L^1 .

1.7 Organization. In Section 2, we introduce the corresponding local weak compactness problem and conclude a local version of weak convergence theorem. Also in this section, we analyze the sequence of metrics near a bubble point via blowing up and conclude a theorem of bubbles on bubbles. This Section is the central piece of this work. In Section 3, we essentially restate the weak compactness theorem in a geometric context. In Section 4, we outline a bubbling procedure and obtain a theorem of bubbles on bubbles.

2 Local problem from an analytic viewpoint

2.1 Introduction

In this section, we consider the problem of weak compactness of a sequence of metrics in a local coordinate disk. In one coordinate chart (D, z) , any metric g is defined as:

$$g = e^{2\varphi}(dx^2 + dy^2), \quad (7)$$

and the curvature function is:

$$K = -\frac{\Delta \varphi}{e^{2\varphi}}. \quad (8)$$

A metric g is said to have a finite area C_1 and a finite energy C_2 if and only if the following conditions are met:

$$\begin{cases} \int_D e^{2\varphi} dx dy & \leq C_1, \\ \int_D \frac{(\Delta \varphi)^2}{e^{2\varphi}} dx dy & \leq C_2. \end{cases} \quad (9)$$

A sequence of metrics $\{g_n\}$ where $g_n = e^{2\varphi_n}(dx^2 + dy^2)$ is said to have finite area C_1 and energy C_2 if and only if each φ_n satisfies the inequality (9). From this point on, in this Section, we will use either $\{\varphi_n\}$ or $\{g_n\}$ to denote a sequence of metrics with finite area C_1 and energy C_2 , unless otherwise specified.

The questions raised in Section 1.1 are : (1) for a sequence of metrics $\{\varphi_n\}$ satisfies the inequalities (9), does this sequence of functions have a uniform bound in $L^\infty(D)$? (2) what is the weak limit of $\{\varphi_n\}$ under some reasonable topology?

Remark 1 *H. Brezis and F. Merle[1] considered a sequence of metrics $\{\varphi_n\}$ satisfies the following equation:*

$$-\Delta\varphi_n = K_n \cdot e^{2\varphi_n}$$

in an open disk D , where $K_n \geq 0$ and $K_n \in L^p(D)$, $e^{2\varphi_n} \in L^{p'}(D)$ where $\frac{1}{p} + \frac{1}{p'} = 1$). They proved that one of the following three alternatives holds true (mutually exclusive):

1. *Vanishing case: $\varphi_n \rightarrow -\infty$ uniformly in any compact subset of D .*
2. *Convergence: there exists a function $\varphi \in H^{2,2}(D)$ such that $\varphi_n \rightharpoonup \varphi$ weakly in $H_{loc}^{2,2}(D)$.*
3. *There exists a finite number of bubble points $\{p_1, p_2, \dots, p_m\}$ such that as a measure,*

$$K_n \cdot e^{2\varphi_n} \rightharpoonup \sum_i \alpha_i \cdot \delta_{p_i}.$$

They conjectured that $\alpha_i = 4\pi \cdot m_i$ for some integer m_i . This conjecture was proved by Y. Y. Li and I. Shafrir[8]. However, it remains open whether m_i actually equals 1.

Our problem differs from the problem consider by H. Brezis and F. Merle significantly. We quote their results here for comparison. The non-compactness occurs in our case is precisely because the curvature changes sign in a small neighborhood and the area element is not in $L^{p'}$ for any $p' > 1$.

For any sub-domain Ω in D , re-label the energy and area for a conformal parameter functions as:

$$A_c(\varphi, \Omega) = \int_{\Omega} e^{2\varphi} dx dy, \quad K_c(\varphi, \Omega) = \int_{\Omega} \frac{(\Delta\varphi)^2}{e^{2\varphi}} dx dy.$$

A “0” metric should have “0” area and energy. Since a “0” metric has a conformal parameters function $-\infty$, we define: $A_c(-\infty, \Omega) = K_c(-\infty, \Omega) = 0$.

For the convenience of notations, we add “ $-\infty$ ” into $H^{2,2}(\Omega)$. The resulted space is denoted by $\hat{H}^{2,2}(\Omega)$. A sequence of functions $\{\varphi_n\} \in H^{2,2}(\Omega)$ weak converges to a function φ_0 in $\hat{H}_{loc}^{2,2}(\Omega)$ if one of the following two alternatives holds true (mutually exclusive):

1. (Vanishing case): If $\varphi_0 \equiv -\infty$, then $\varphi_n \rightarrow -\infty$ uniformly in any compact sub-domain of Ω .
2. (Non-vanishing case): If $\varphi_0 \in H^{2,2}(\Omega)$, then $\varphi_n \rightharpoonup \varphi_0$ weakly in $H_{loc}^{2,2}(\Omega)$.

Definition 1 A point p is said to be a bubble point of $\{\varphi_i\}$ if for any $r > 0$,

$$\lim_{n \rightarrow \infty} \int_{D_r(p)} \frac{(\Delta \varphi_n)^2}{e^{2\varphi_n}} dx dy \geq \alpha > 0, \quad \lim_{n \rightarrow \infty} \int_{D_r(p)} e^{2\varphi_n} dx dy \geq \beta > 0. \quad (10)$$

where $D_r(p)$ denotes a coordinate disk centered at p with radius r . The largest possible numbers α and β are the concentration weights of the energy function and area function at this point p .

Clearly, if p is a bubble point of $\{\varphi_n\}$, then p is a bubble point of any subsequence of $\{\varphi_n\}$.

Example 1. Let $g_n = \frac{n^2}{(1+n^2|z+n^{-0.33}|^2)^2} |dz|^2$ be a sequence of metrics in S^2 with a constant curvature of 1. This sequence of metrics then converges to 0 at every point (including the point $z = 0$) on S^2 . However, the concentrations of energy and area at $z = 0$ are $4\pi, 4\pi$. The metrics could be renormalized as: $\tilde{g}_n(z) = g_n((z - n^{-0.33})/n)$. This new sequence \tilde{g}_n weakly converges to a metric in S^2 with constant curvature.

The main theorems in this Section are:

Theorem 1 Let $\{\varphi_n, n \in \mathbf{N}\}$ be a sequence of metrics in $H^{2,2}(D)$ with a finite area C_1 and energy C_2 . There exists a subsequence $\{\varphi_{n_j}, j \in \mathbf{N}\}$ of $\{\varphi_n\}$, a finite number of bubble points $\{p_1, p_2, \dots, p_m\}$ ($0 \leq m \leq \sqrt{\frac{C_1 \cdot C_2}{4\pi^2}}$) with respect to $\{\varphi_{n_j}, j \in \mathbf{N}\}$, and a metric $\varphi_0 \in \hat{H}_{loc}^{2,2}(D \setminus \{p_1, p_2, \dots, p_m\})$ such that:

$$\varphi_{n_j} \rightharpoonup \varphi_0 \text{ in } \hat{H}_{loc}^{2,2}(D \setminus \{p_1, p_2, \dots, p_m\}).$$

If the energy and area concentrations in each bubble point p_i are A_{p_i} and K_{p_i} for any $i \in [1, m]$, then:

$$\lim_{j \rightarrow \infty} A_c(\varphi_{n_j}, D) = A_c(\varphi_0, D \setminus \{p_1, p_2, \dots, p_m\}) + \sum_{i=1}^m A_{p_i} \quad (11)$$

$$\lim_{j \rightarrow \infty} K_c(\varphi_{n_j}, D) \geq K_c(\varphi_0, D \setminus \{p_1, p_2, \dots, p_m\}) + \sum_{i=1}^m K_{p_i}. \quad (12)$$

Remark 2 The equality in formula 12 holds if $\{\varphi_n\}$ minimizes the energy function.

Theorem 2 For any metric φ with a finite area C_1 and energy C_2 in $D \setminus \{0\}$, define $\phi(r) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(r \cos \theta, r \sin \theta) d\theta$. The following three statements hold true:

1. $\lim_{r \rightarrow 0} (\varphi(r \cos \theta, r \sin \theta) + \ln r) = -\infty$.
2. $\lim_{r \rightarrow 0} \phi'_r(r) \cdot r$ exists and is finite.
3. There exists a constant $\beta \in (0, 1)$ and two constants C_3, C_4 such that:

$$\frac{1}{\beta}(\phi(r) + \ln r) + C_3 \leq \varphi(r \cos \theta, r \sin \theta) + \ln r \leq \beta(\phi(r) + \ln r) + C_4.$$

Theorem 3 (Bubbles on bubbles). Let $\{\varphi_n\}$ be a sequence of metrics in D with finite area C_1 and finite energy C_2 . Suppose that $p = 0$ is the only bubble point in D with area concentration A_p and energy concentration K_p . Suppose there exists a metric $\varphi_0 \in \hat{H}^{2,2}(D \setminus \{p\})$ such that $\varphi_n \rightharpoonup \varphi_0$ in $\hat{H}_{loc}^{2,2}(D \setminus \{p\})$. A sequence of numbers $\{\epsilon_n \searrow 0\}$ can be chosen to re-normalize the sequence of metrics as: $\phi_n(x, y) = \varphi_n(\epsilon_n \cdot x, \epsilon_n \cdot y) + \ln \epsilon_n (\forall n \in \mathbf{N})$. There exists a subsequence $\{\varphi_{n_j}, j \in \mathbf{N}\}$ of $\{\varphi_n\}$, a finite number of bubble points $\{q_1, q_2, \dots, q_m\} (0 \leq m \leq \sqrt{\frac{A_p \cdot K_p}{4\pi^2}})$ with respect to the subsequence of metrics $\{\phi_{n_j}\}$, a metric $\phi_0 \in \hat{H}^{2,2}(S^2 \setminus \{\infty, q_1, q_2, \dots, q_m\})$ such that:

$$\phi_{n_j} \rightharpoonup \phi_0 \text{ in } \hat{H}_{loc}^{2,2}(S^2 \setminus \{\infty, q_1, q_2, \dots, q_m\}).$$

If the energy and area concentrations of $\{\phi_n\}$ at each point q_i are K_{q_i} and A_{q_i} , then:

$$A_p \geq A_c(\phi_0, S^2 \setminus \{q_1, q_2, \dots, q_m\}) + \sum_{i=1}^m A_{q_i} \quad (13)$$

$$K_p \geq K_c(\phi_0, S^2 \setminus \{q_1, q_2, \dots, q_m\}) + \sum_{i=1}^m K_{q_i}. \quad (14)$$

If $\phi_0 \equiv -\infty$ (vanishing case), then $m \geq 2$ and $p(z = 0)$ is a bubble point of $\{\phi_{n_j}, j \in \mathbf{N}\}$.

Remark 3 The difference of the left side and right side of the inequality 13 represents the amount of area lost during the blowing up procedure. If this amount is zero, there is no area trapped in the neck.

In Subsection 3.2, we prove three important lemmas (lemma 2,4 and 6), which provide a technical foundation for the main theorems. The proof are rather technical, readers are then encouraged to skip Subsection 3.2 and read the other Subsections first. In Subsection 3.3, we prove a weak convergence theorem. In Subsection 3.4, we briefly describe the properties of the limit metrics. In Subsection 3.5, we show that a renormalized sequence of metrics at each bubble point will have a weak convergent subsequence.

2.2 Small energy lemmas

In this subsection, the notion of a pseudo bubble point is introduced. It is subsequently used to prove three key lemmas: lemma 2, 4 and 6. Lemma 2 shows that the concentration of total energy (product of curvature energy and area) at each bubble point must be greater than $4\pi^2$. Thus, there are at most a finite number of bubble points for any subsequence of metrics. Lemma 4 shows that if a point is not a pseudo bubble point, then the sequence of metrics in a neighborhood of that point is uniformly bounded from above. Lemma 6 shows that in any domain, if the metrics are uniformly bounded from above, then either the sequence of metrics approaches 0 everywhere in its domain, or a subsequence of these metrics weakly converges in $H^{2,2}$ in any compact sub-domain.

For any $p \in D$, a small disk center at p with radius r will be denoted by $D_r(p)$.

$$D_r(p) = \{(x, y) \in D | (x - x_p)^2 + (y - y_p)^2 < r^2\}.$$

Define local energy and area functions with respect to any point $p \in D$ as the following:

$$\begin{aligned} K_p(r) &= \overline{\lim}_{k \rightarrow \infty} \int_{D_r(p)} \frac{(\Delta \varphi_k)^2}{e^{2\varphi_k}} dx dy, \quad \forall r > 0, \\ A_p(r) &= \overline{\lim}_{k \rightarrow \infty} \int_{D_r(p)} e^{2\varphi_k} dx dy, \quad \forall r > 0. \end{aligned}$$

In this definition, the limit taken is only an upper limit, since it is not known whether $\{\varphi_n\}$ has any weak convergent subsequence.

Definition 2 *The energy and area concentration functions of a sequence of metrics $\{g_k\}$ at any point $p \in M$, are defined as follows:*

$$K_p = \lim_{r \rightarrow 0} K_p(r), \quad A_p = \lim_{r \rightarrow 0} A_p(r).$$

Any point $p \in D$ is called a pseudo bubble point if and only if $A_p > 0$ and $E_p > 0$. Later, we could show that $A_p > 0$ actually implies $E_p > 0$. At a pseudo bubble point, there exists a subsequence of $\{\varphi_n\}$ such that this subsequence has a positive amount of area and energy concentrations there. If we pass to this subsequence, the pseudo bubble point becomes a real bubble point.

Proposition 1 *Let p be a pseudo bubble point of a sequence of metrics $\{\varphi_n, n \in \mathbb{N}\}$, there then exists a subsequence of $\{\varphi_n\}$ such that p is a real bubble point with respect to this subsequence.*

Proof. The proof is straightforward.

Definition 3 *The waist concentration function, $l_p(\rho, \rho_0)$, for any $0 < \rho < \rho_0$ is defined as:*

$$l_p(\rho, \rho_0) = \varliminf_{n \rightarrow \infty} \min_{\rho \leq r \leq \rho_0} |\partial D_r|_{g_n} = \varliminf_{n \rightarrow \infty} \min_{\rho \leq r \leq \rho_0} \int_0^{2\pi} e^{\varphi_n(r \cos \theta, r \sin \theta)} r d\theta.$$

Lemma 1 *Let $\{\varphi_n\}$ be a sequence of metrics with finite area C_1 and finite energy C_2 . For any $\rho_0 > 0$, we have $\lim_{\rho \rightarrow 0} l_p(\rho, \rho_0) = 0$.*

Proof. If the lemma is false, then there exists a number $\epsilon > 0$ such that: $\lim_{\rho \rightarrow 0} l_p(\rho, \rho_0) = 2\epsilon > 0$. Choose ρ small enough so that:

$$\frac{\epsilon^2}{2\pi} \ln \frac{\rho_0}{\rho} > 2 \cdot C_1. \quad (15)$$

Since $l_p(\rho, \rho_0)$ is a monotonely increasing function on its variable $\rho > 0$,

$$l_p(\rho, \rho_0) \geq 2\epsilon, \quad \forall \quad 0 < \rho < \rho_0.$$

In other words,

$$\varliminf_{n \rightarrow \infty} \min_{\rho \leq r \leq \rho_0} |\partial D_r|_{g_n} \geq 2\epsilon.$$

Fixing the pair of numbers ρ, ρ_0 , there then exists a number n_0 such that $|\partial D_r|_{g_{n_0}} > \epsilon, \quad \forall \quad \rho \leq r \leq \rho_0$. In a local coordinate,

$$|\partial D_r|_{g_{n_0}} = \int_0^{2\pi} e^{\varphi_{n_0}} r d\theta > \epsilon, \quad \forall r \in [\rho, \rho_0].$$

However,

$$\begin{aligned} 2\pi \cdot C_1 &\geq \int_{\rho}^{\rho_0} \int_0^{2\pi} e^{2\varphi_{n_0}} \cdot r \cdot d\theta dr \cdot \int_0^{2\pi} 1 d\theta \\ &\geq \int_{\rho}^{\rho_0} \left(\int_0^{2\pi} e^{\varphi_{n_0}} d\theta \right)^2 r dr \\ &\geq \int_{\rho}^{\rho_0} \left(\frac{\epsilon}{r} \right)^2 r dr \\ &\geq \epsilon^2 \ln \frac{\rho_0}{\rho} > 4\pi \cdot C_1. \end{aligned}$$

The last inequality holds true because of inequality (15). Thus, $2\pi > 4\pi$, which is impossible. The lemma is then proved. QED.

The following theorem is a generalization of the classical isoperimetric inequality. It is a key theorem which we will use it over and over again.

Theorem 4 (Readers are referred to [4] for further reference). Let g be a metric in an Euclidean disk D such that $\int_D |K_g| dg < \infty$. For any disk $D_1 \subset\subset D$, we have:

$$\int_{D_1} |K_g| dg \geq 2\pi - \frac{(\int_{\partial D_1} ds_g)^2}{2 \int_{D_1} dg} = 2\pi - \frac{|\partial D_1|_g^2}{2 \int_{D_1} dg}.$$

Lemma 2 Let $\{\varphi_n\}$ be a sequence of metrics with finite area C_1 and finite energy C_2 . If p is a bubble point of $\{\varphi_k\}$, then the following inequality holds true:

$$\sqrt{K_p \cdot A_p} \geq 2\pi.$$

Remark 4 (a) The best constant in the above estimate is 4π .
(b) This lemma also proves that $A_p > 0$ if and only if $K_p > 0$.

This lemma implies that there are only a finite number of bubble points. It can also be regarded as a “small energy lemma.” In other words, if the total energy $\sqrt{K(\Omega) \cdot A(\Omega)}$ is small enough ($\leq 2\pi$), any weak convergent subsequence of metrics does not have any bubble point in any compact sub-domain of Ω .

Proof. Suppose p is a bubble point and $A_p > 0$. Let $\epsilon > 0$ be any small positive number. Recalled that $A_p = \lim_{r \rightarrow 0} A_p(r)$. Since $A_p(r)$ is a monotonely increased function on its variable r , then $\lim_{r \rightarrow 0} A_p(r) \geq A_p > 0$. Choose ρ_0 and for n large enough:

$$A_p \leq A_{\rho_0}(r) = \overline{\lim}_{n \rightarrow \infty} \int_{D_{\rho_0}} dg_n < (1 + \frac{\epsilon}{2})A(p).$$

For n large enough, we have

$$\int_{D_{\rho_0}} dg_n < (1 + \epsilon)A(p)$$

Lemma 1 then implies:

$$\lim_{\rho \rightarrow 0} l_p(\rho, \rho_0) = 0, \quad \forall \rho_0 > 0.$$

For any $\epsilon > 0$, we choose a small number $\rho_1 < \rho_0$ such that $l_p(\rho_1, \rho_0) < \epsilon$. There exists a positive number N which depends only on ϵ such that (after passing to a subsequence):

$$\min_{\rho_1 \leq r \leq \rho_0} |\partial D_r|_{g_n} < 2\epsilon, \quad \forall n > N. \quad (16)$$

There exists a number $\rho_n \in [\rho_1, \rho_0]$ such that:

$$|\partial D_{\rho_n}|_{g_n} < 3\epsilon, \quad \forall \rho_1 \leq \rho_n \leq \rho_0.$$

Therefore,

$$A_p \leq \int_{D_{\rho_1}} dg_n \leq \int_{D_{\rho_n}} dg_n \leq (1 + \epsilon)A(p).$$

According to Theorem 4, we have:

$$\begin{aligned} \int_{D_{\rho_n}} |K_{g_n}| dg_n &\geq 2\pi - \frac{|\partial D_{\rho_n}|_{g_n}^2}{2 \int_{D_{\rho_n}} dg_n} \\ &> 2\pi - \frac{9\epsilon^2}{2A(p)} > 0. \end{aligned}$$

The last inequality holds for any small $\epsilon > 0$. Hence, we have:

$$\begin{aligned} \int_{D_{\rho_n}} K_{g_n}^2 dg_n &\geq \frac{(\int_{D_{\rho_n}} K_{g_n} dg_n)^2}{\int_{D_{\rho_n}} dg_n} \\ &> \frac{(2\pi - \frac{9\epsilon^2}{2A(p)})^2}{(1 + \epsilon)A(p)}. \end{aligned}$$

Since $\rho_n < \rho_0$,

$$\int_{D_{\rho_0}} K_{g_n}^2 dg_n > \frac{(2\pi - \frac{9\epsilon^2}{2A(p)})^2}{(1 + \epsilon)A(p)}, \quad \forall n > N.$$

In other words,

$$\underline{\lim}_{n \rightarrow \infty} \int_{D_{\rho_0}} K_{g_n}^2 dg_n > \frac{(2\pi - \frac{9\epsilon^2}{2A(p)})^2}{(1 + \epsilon)A(p)}.$$

Let $\epsilon \rightarrow 0$, then let $\rho_0 \rightarrow 0$, we have:

$$K_p = \lim_{\rho_0 \rightarrow 0} \underline{\lim}_{n \rightarrow \infty} \int_{D_{\rho_0}} K_{g_n}^2 dg_n \geq \frac{4\pi^2}{A(p)}.$$

The lemma is then established. QED.

Lemma 3 *If $\{\varphi_n\}$ has finite area C_1 , then $\underline{\lim}_{n \rightarrow \infty} \min_{0 \leq \rho \leq r \leq \rho_0} \frac{1}{2\pi} \int_0^{2\pi} \varphi_n(r \cos \theta, r \sin \theta) d\theta$ is bounded from above for any interval $[\rho, \rho_0]$.*

Proof. If this lemma is false, there then exists $0 < \rho < \rho_0$ such that:

$$\lim_{n \rightarrow \infty} \min_{\rho \leq r \leq \rho_0} \frac{1}{2\pi} \int_0^{2\pi} \varphi_n(r \cos \theta, r \sin \theta) d\theta = \infty.$$

Passing to a subsequence if necessary, we may assume:

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi_n(r \cos \theta, r \sin \theta) d\theta > n, \quad \forall r \in [\rho, \rho_0].$$

A Schwartz type of inequality implies:

$$\frac{1}{2\pi} \int_0^{2\pi} e^{2\varphi_n(r \cos \theta, r \sin \theta)} d\theta > e^{\frac{1}{2\pi} \int_0^{2\pi} 2\varphi_n(r \cos \theta, r \sin \theta) d\theta} > e^{2n}, \quad \forall r \in [\rho, \rho_0].$$

The last inequality implies:

$$C_1 > \frac{1}{2\pi} \int_\rho^{\rho_0} \int_0^{2\pi} e^{2\varphi_n(r \cos \theta, r \sin \theta)} r dr d\theta > e^{2n} \int_\rho^{\rho_0} r dr = e^{2n} (\rho_0^2 - \rho^2)/2 \rightarrow \infty.$$

This is a contradiction. This lemma is proved. QED.

The following lemma shows that the conformal parameters $\{\varphi_k\}$ must have a uniform upper bound, away from the set of bubble points.

Lemma 4 *Let $\{\varphi_n\}$ be a sequence of metrics with finite area C_1 and finite energy C_2 . If p is not a pseudo bubble point of $\{\varphi_k, k \in \mathbf{N}\}$, i.e., $A(p) = 0$, there then exists a small neighborhood $\mathcal{O}(p)$ of p and a positive constant C such that $\sup_{k \in \mathbf{N}} \sup_{q \in \mathcal{O}(p)} \varphi_k(q) \leq C$.*

Proof. Define a new function:

$$A_n(\rho) = \int_{D_\rho} dg_n = \int_{x^2+y^2 \leq \rho^2} e^{2\varphi_n} dx dy, \quad \forall n \in \mathbf{N}, \quad \forall \rho > 0.$$

Choose a small coordinate disk $D_{r_0}(0)$ so that: $2 \cdot C_2 \cdot A_n(r_0) < \pi^2$. If this lemma is false, we could modified the sequence of metrics slightly so that: $\varphi_n(p) \rightarrow \infty$. We want to draw a contradiction from this assumption.

For any pair of numbers $r_1 > r_2$, consider the following:

$$\begin{aligned} & \left| \int_0^{2\pi} \frac{\partial \varphi_n(r_1 \cos \theta, r_1 \sin \theta)}{\partial r} r_1 d\theta - \int_0^{2\pi} \frac{\partial \varphi_n(r_2 \cos \theta, r_2 \sin \theta)}{\partial r} r_2 d\theta \right| \\ &= \left| \int_{r_2}^{r_1} \int_0^{2\pi} \frac{\partial}{\partial r} \left(\frac{\partial \varphi_n(r \cos \theta, r \sin \theta)}{\partial r} \cdot r \right) d\theta dr \right| = \left| \int_{r_2}^{r_1} \int_0^{2\pi} (\varphi_n'' \cdot r + \varphi_n') d\theta dr \right| \\ &= \left| \int_{r_2}^{r_1} \int_0^{2\pi} \Delta \varphi_n(r \cos \theta, r \sin \theta) r d\theta dr \right| \\ &\leq \left(\int_{r_2}^{r_1} \int_0^{2\pi} \frac{(\Delta \varphi_n)^2}{e^{2\varphi_n}} r d\theta dr \right)^{\frac{1}{2}} \left(\int_{r_2}^{r_1} \int_0^{2\pi} e^{2\varphi_n} r d\theta dr \right)^{\frac{1}{2}}. \end{aligned}$$

Since the energy of this sequence of metrics is uniformly bounded from above, the previous inequality implies:

$$\left| \int_0^{2\pi} \frac{\partial \varphi_n(r_1 \cos \theta, r_1 \sin \theta)}{\partial r} r_1 d\theta - \int_0^{2\pi} \frac{\partial \varphi_n(r_2 \cos \theta, r_2 \sin \theta)}{\partial r} r_2 d\theta \right| \leq C_2^{\frac{1}{2}} \cdot \left(\int_{r_2}^{r_1} \int_0^{2\pi} e^{2\varphi_n} r d\theta dr \right)^{\frac{1}{2}}.$$

Fixing the number n , observe that $\lim_{r \rightarrow 0} \int_0^{2\pi} \frac{\partial \varphi_n(r \cos \theta, r \sin \theta)}{\partial r} r d\theta = 0$. Let $r_2 \rightarrow 0$ and $r_1 = r$, the result is:

$$\left| \int_0^{2\pi} \frac{\partial \varphi_n(r \cos \theta, r \sin \theta)}{\partial r} r d\theta \right| < C_2^{\frac{1}{2}} \cdot \sqrt{A_n(r)}.$$

Define $\psi_n(r) = \frac{1}{2\pi} \int_0^{2\pi} \varphi_n(r \cos \theta, r \sin \theta) d\theta$, then:

$$\begin{aligned} |\psi_n(r) - \psi_n(0)| &\leq \int_0^r \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{\partial \varphi_n}{\partial \rho} \rho d\theta \right| \frac{d\rho}{\rho} \\ &\leq \int_0^r \frac{1}{2\pi} \cdot C_2^{\frac{1}{2}} \cdot \sqrt{A_n(\rho)} \cdot \frac{d\rho}{\rho}. \end{aligned}$$

Since $\psi_n(0) = \varphi_n(p)$, therefore $(0 < \alpha < 1)$

$$|\psi_n(r) - \varphi_n(p)| \leq \frac{1}{2\pi} \cdot C_2^{\frac{1}{2}} \cdot \int_0^r \left(\frac{A_n(\rho)}{\rho^\alpha} \right)^{\frac{1}{2}} \cdot \frac{\rho^{\frac{\alpha}{2}}}{\rho} d\rho. \quad (17)$$

Following lemma 3, $\lim_{n \rightarrow \infty} \psi_n(r) < \infty$. It is a contradiction if the right hand side (RHS) of the previous inequality (17) is uniformly bounded from above since $\{\varphi_n(p)\} \rightarrow \infty$. However, the (RHS) of the inequality (17) is bounded according to the next lemma (choose $\alpha = 1$). The lemma is then proved. QED.

Lemma 5 *Let $\{\varphi_n\}$ be a sequence of metrics with finite area C_1 and finite energy C_2 . Suppose $A_p = 0$. For any small number $r > 0$, there exists a positive constant C and a number N such that $(0 < \alpha < 2)$:*

$$\frac{\int_{D_\rho} e^{2\varphi_n} dx dy}{\rho^\alpha} = \frac{A_n(\rho)}{\rho^\alpha} < C,$$

if n is large enough.

Proof. Choose a small coordinate disk $D_r(0)$ so that: $2 \cdot C_2 \cdot A_n(r) < \pi^2(2 - \alpha)^2$ (since $A_p = 0$). Let C be any number large enough such that:

$$\frac{A_n(r)}{r^\alpha} < C, \forall n \in \mathbf{N}.$$

It is claimed that this lemma holds true for this constant C . Otherwise, there exists a number $\rho_n < r$, such that $A_n(\rho_n) - C \cdot \rho_n^\alpha > 0$. Consider the function $F_n(\rho) = A_n(\rho) - C \cdot \rho^\alpha$. We have $F_n(r) < 0 < F_n(\rho_n)$. There then exists an interior point $r_n \in (\rho_n, r_0)$ such that:

$$F_n(r_n) = 0, \quad F'_n(r_n) < 0,$$

or,

$$A_n(r_n) = C \cdot r_n^\alpha, \quad \int_0^{2\pi} e^{2\varphi_n} r_n d\theta < C \cdot \alpha \cdot r_n^{\alpha-1}.$$

Using a Schwartz type inequality, we have:

$$\left(\int_0^{2\pi} e^{\varphi_n} r_n d\theta\right)^2 < \int_0^{2\pi} e^{2\varphi_n} r_n d\theta \cdot \int_0^{2\pi} r_n d\theta < 2\pi \cdot C \cdot \alpha \cdot r_n^\alpha.$$

In other words, $|\partial D_{r_n}|_{g_n}^2 < 2\pi\alpha \cdot C \cdot r_n^\alpha$. Therefore,

$$\begin{aligned} \int_{D_{r_n}} |K_{g_n}| d g_n &> 2\pi - \frac{|\partial D_{r_n}|_{g_n}^2}{2A_{r_n}} \\ &> 2\pi - \frac{2\pi\alpha \cdot C \cdot r_n^\alpha}{2C \cdot r_n^\alpha} \\ &= \pi(2 - \alpha) > 0. \end{aligned}$$

Using a Schwartz type inequality again, we have:

$$\int_{D_{r_n}} K_{g_n}^2 d g_n > \frac{(\pi(2 - \alpha))^2}{A_n(r_n)} \geq \frac{(\pi(2 - \alpha))^2}{A_n(r)}.$$

Thus,

$$C_2 \geq \int_{D_r} K_{g_n}^2 d g_n \geq \int_{D_{r_n}} K_{g_n}^2 d g > \frac{\pi^2(2 - \alpha)^2}{A_n(r)} > 2 \cdot C_2,$$

which is a contradiction. The lemma is then proved. QED.

Lemma 6 *Let $\{\varphi_n\}$ be a sequence of metrics with finite area C_1 and finite energy C_2 . Suppose $\sup_{k \in \mathbf{N}} \max_{q \in D} \varphi_k(q) < C_3$. Let $\Omega \subset D$ be any compact sub-domain of D . There exists a constant $\beta \in (0, 1)$ which depends only on C_1, C_2, C_3 , and the domains Ω, D such that:*

$$\sup_{\Omega} \varphi_k \leq \beta \cdot \inf_{\Omega} \varphi_k + C, \quad \forall k \in \mathbf{N}.$$

In particular, either $\{\varphi_k\}$ vanishes everywhere on D or there exists a constant C such that:

$$\inf_{q \in \Omega} \varphi_k(q) > -C, \quad \forall k \in \mathbf{N}.$$

Proof. The conditions in this lemma are:

$$\begin{cases} \sup_{q \in D} \varphi_n(q) &< C_3, \\ \int_D \frac{(\Delta \varphi_n)^2}{e^{2\varphi_n}} dx dy &< C_2, \\ \int_D e^{2\varphi_n} dx dy &< C_1 \end{cases}$$

From the first two inequalities, we imply:

$$\|\Delta \varphi_n\|_{L^2(D)} = \int_D (\Delta \varphi_n)^2 dx dy < C.$$

Decompose the conformal parameter functions φ_n as $\varphi_n = u_n + v_n$, where u_n, v_n satisfy the following:

$$\begin{cases} \Delta u_n &= \Delta \varphi_n, \\ u_n|_{\partial D} &= 0; \end{cases}$$

and

$$\begin{cases} \Delta v_n &= 0, \\ v_n|_{\partial D} &= \varphi_n|_{\partial D}. \end{cases}$$

Clearly $\|u_n\|_{H^{2,2}(D)} < C$. This implies that $\max_{p \in D} |u_n(p)| < C$, $\forall n \in \mathbf{N}$.

Since φ_n is bounded from above by the initial assumption, the harmonic functions $v_n = \varphi_n - u_n$ is bounded from above. For any sub-domain $\Omega \subset D$, there exists a constant $\beta \in (0, 1)$ such that $\sup_{\Omega} (C - v_n) \leq \frac{1}{\beta} \cdot \inf_{\Omega} (C - v_n)$. Thus,

$$\sup_{\Omega} \varphi_n \leq \beta \cdot \inf_{\Omega} \varphi_n + C.$$

QED.

2.3 Locally weakly convergence

Proposition 2 *Let $\{\varphi_k, k \in \mathbf{N}\}$ be a sequence of metrics in D with finite area C_1 and energy C_2 . There exists at most a finite number of bubble points (bounded by $\sqrt{\frac{C_1 \cdot C_2}{4\pi^2}}$) in D for any subsequence of metrics of $\{\varphi_k\}$. Moreover, there exists a subsequence of $\{\varphi_k\}$ which has a finite number of bubble points and has no additional pseudo bubble points in D .*

Proof: We first prove that there exists at most a finite number of bubble points for any sequence of metrics which satisfies inequality 9 uniformly. Suppose that p_1, p_2, \dots, p_k are all of the bubble points. On one hand, we have:

$$\sum_{i=1}^k A(p_i) \leq \int_D e^{2\varphi_n} dx dy \leq C_1.$$

On the other hand, lemma 2 implies:

$$K(p_i) \geq \frac{4\pi^2}{A(p_i)}, \quad \forall i = 1, 2, \dots, k.$$

The total concentrated energy of this sequence of metrics at these bubble points must be less than the total amount of energy of this sequence of metrics. Thus,

$$\begin{aligned} C_2 &\geq \int_D \frac{(\Delta \varphi_n)^2}{e^{2\varphi_n}} dx dy \\ &\geq \sum_{i=1}^k K(p_i) \\ &\geq \sum_{i=1}^k \frac{4\pi^2}{A(p_i)} \\ &\geq \frac{4(k\pi)^2}{\sum_{i=1}^k A(p_i)} \geq \frac{4(k\pi)^2}{C_1}. \end{aligned}$$

Therefore, $k \leq \sqrt{\frac{C_1 \cdot C_2}{4\pi^2}}$.

Suppose the original sequence of metrics has l distinct bubble points and has at least one additional pseudo point p . Passing to an appropriate subsequence, (by proposition 1), p is then a bubble point for this subsequence. This subsequence then has $(l+1)$ distinctive bubble points. It is claimed that a subsequence of $\{\varphi_n\}$ can be selected so that it has only a finite number of bubble points and it has no additional pseudo bubble points. Otherwise, we can keep passing to an appropriate subsequence to convert any additional pseudo bubble point into a new bubble point. Eventually, we will obtain a subsequence of metrics in D which has more than $\sqrt{\frac{C_1 \cdot C_2}{4\pi^2}}$ number of bubble points. This is a contradiction. The proposition is then proved. QED.

Proof of Theorem 1. Passing to a subsequence of $\{\varphi_n\}$ if necessary, so that $\{\varphi_n\}$ has exactly $m(\geq 0)$ number of bubble points and has no additional pseudo bubble points. Denote these bubble points by $\{p_1, p_2, \dots, p_m\}$. Choose two compact sub-domains D^1 and D^2 so that:

$$\{p_1, p_2, \dots, p_m\} \subset D^1 \subset D^2 \subset\subset D.$$

Let $\epsilon > 0$ be small enough so that $\{D_\epsilon(p_s), 1 \leq s \leq m\}$ are disjoint disks in D_1 . Let $D_{i,j}$ denote the following domains (see Figure 3 below):

$$D_{i,j} = D^i \setminus \left(\bigcup_{1 \leq s \leq m} D_{\frac{1}{2^j}\epsilon}(p_s) \right), \quad i = 1, 2, \forall j \in \mathbf{N}.$$

Clearly, $D_{1,j}$ is a compact sub-domain of $D_{2,j+1}$. Fixing a number j , there exists a constant c_j independent of $\{\varphi_n\}$ such that:

$$\varphi_n(p) \leq c_j, \quad \forall p \in D_{2,j+1}, \forall n \in \mathbf{N}. \quad (18)$$

If not, there exists a sequence of points $q_k \in D_{2,j+1}$ such that:

$$\lim_{k \rightarrow \infty} \varphi_k(q_k) = \infty. \quad (19)$$

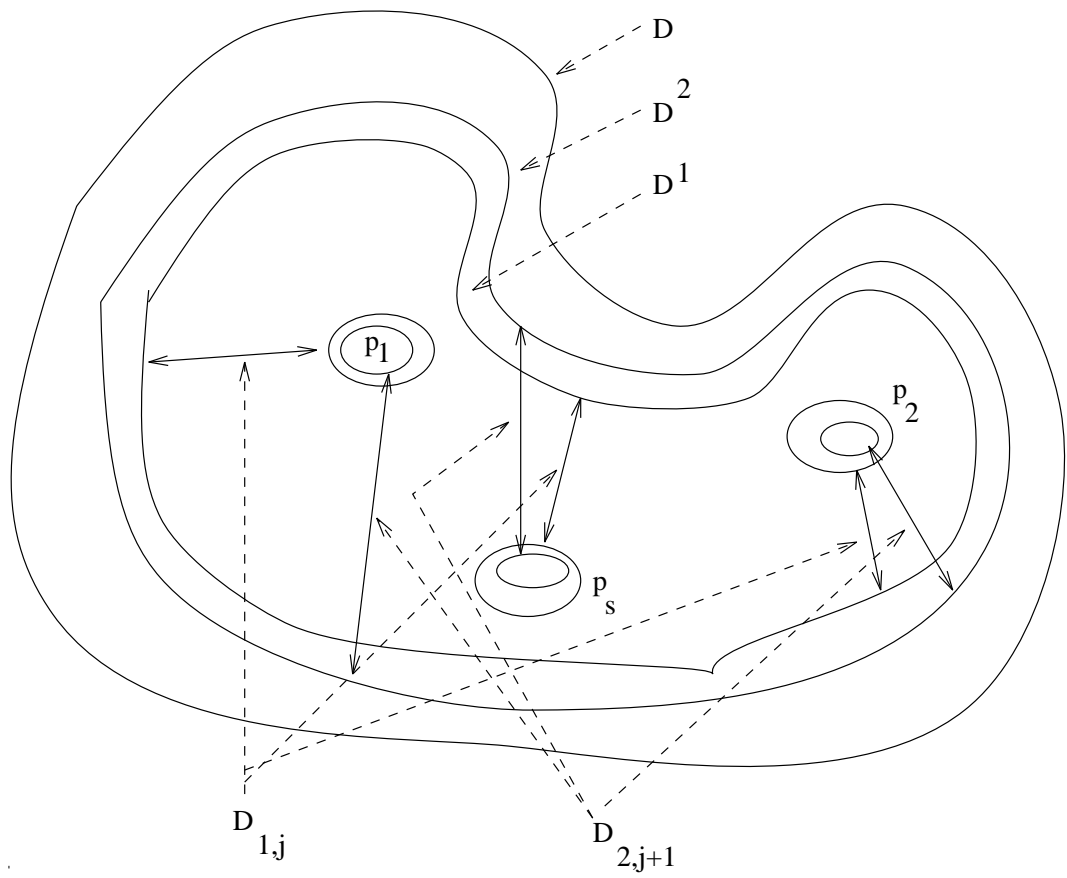


Figure 3: Compact sub-domains

Consider a cluster point $q \in \overline{D_{2,j+1}}$ of $\{q_k\}$ such that $q_k \rightarrow q$ (passing to a subsequence of $\{q_k\}$ if necessary). According to the initial assumption, q is not a pseudo bubble point of $\{\varphi_n\}$. Lemma 4 then implies that there exists a constant C and an open neighborhood \mathcal{O} of p such that $\sup_n \sup_{q \in \mathcal{O}} \varphi_n(q) < C$. This contradicts with equation (19). Therefore, the inequality (18) holds true. Thus,

$$\int_{D_{2,j}} (\Delta \varphi_n)^2 dx dy \leq e^{2c_j} \cdot C_2.$$

According to lemma 6, either $\varphi_n \rightarrow -\infty$ in $D_{2,j+1}$ or there exists another constant c'_j such that

$$\varphi_n(p) \geq -c'_j, \quad \forall p \in D_{1,j}, \forall n \in \mathbf{N}.$$

If $\varphi_n \rightarrow -\infty$ in $D_{1,j}$, define $\varphi_{0,j} \equiv -\infty$. If $\varphi_n \not\rightarrow -\infty$ in $D_{1,j}$, then φ_n are uniformly bounded in $H^{2,2}(D_{1,j})$. There then exists a function $\varphi_{0,j} \in H^{2,2}(D_{1,j})$ such that $\varphi_n \rightharpoonup \varphi_{0,j}$ in $H_{loc}^{2,2}(D_{1,j})$. Thus, in either case, we have:

$$\varphi_n \rightharpoonup \varphi_{0,j} \text{ in } \hat{H}_{loc}^{2,2}(D_{1,j}).$$

Define $\{\varphi_{0,j}, j \in \mathbf{N}\}$ successively in $D_{1,j}$ for $j = 1, 2, \dots$ such that:

$$\varphi_{n,j} \rightharpoonup \varphi_{0,j} \text{ in } \hat{H}_{loc}^{2,2}(D_{1,j}),$$

where $\{\varphi_{n,j}\} (j > 1)$ is a subsequence of $\{\varphi_{n,j-1}\}$. Consider the diagonal subsequence $\{\varphi_{n,n}\}$. For any fixed $j > 0$, we have:

$$\varphi_{n,n} \rightharpoonup \varphi_{0,j} \text{ in } \hat{H}_{loc}^{2,2}(D_{1,j}).$$

Clearly, for any $i > j$, we have $\varphi_{0,i} \equiv \varphi_{0,j}$ in $D_{1,j}$. In particular, $\varphi_{0,i} \equiv -\infty$ if and only if $\varphi_{0,j} \equiv -\infty$ (lemma 6). Thus, $\{\varphi_{0,j}, j \in \mathbf{N}\}$ defines a metric φ_0 in $\hat{H}^{2,2}(D_1 \setminus \{p_1, p_2, \dots, p_m\})$ by

$$\varphi_0(p) = \varphi_{0,j}(p), \quad \forall p \in D_1 \setminus \{p_1, p_2, \dots, p_m\}.$$

Therefore,

$$\varphi_{n,n} \rightharpoonup \varphi_0 \text{ in } \hat{H}_{loc}^{2,2}(D_1 \setminus \{p_1, p_2, \dots, p_m\}).$$

Denote $\{\varphi_{n,n}\}$ by $\{\varphi_n\}$ and define

$$A_s(j) = \lim_{n \rightarrow \infty} A_c(\varphi_n, D_{\frac{1}{2^j}\epsilon}(p_s)), \quad K_s(j) = \lim_{n \rightarrow \infty} K(\varphi_n, D_{\frac{1}{2^j}\epsilon}(p_s)), \quad \forall 1 \leq s \leq m.$$

Then,

$$A_{p_s} = \lim_{j \rightarrow \infty} A_s(j), \quad K_{p_s} = \lim_{j \rightarrow \infty} K_s(j).$$

In $D_{1,j}$, we have:

$$\lim_{n \rightarrow \infty} A_c(\varphi_n, D_{1,j}) = A_c(\varphi_0, D_{1,j}), \quad \lim_{n \rightarrow \infty} K_c(\varphi_n, D_{1,j}) \geq K_c(\varphi_0, D_{1,j}).$$

Therefore,

$$\begin{aligned}\lim_{n \rightarrow \infty} A_c(\varphi_n, D_1) &= A_c(\varphi_0, D_{1,j}) + \sum_{s=1}^m A_s(j), \\ \lim_{n \rightarrow \infty} K_c(\varphi_n, D_1) &\geq K_c(\varphi_0, D_{1,j}) + \sum_{s=1}^m K_s(j).\end{aligned}$$

Taking limit as $j \rightarrow \infty$, we have

$$\begin{aligned}\lim_{n \rightarrow \infty} A_c(\varphi_n, D_1) &= A_c(\varphi_0, D_1 \setminus \{p_1, p_2, \dots, p_m\}) + \sum_{s=1}^m A_{p_s}, \\ \lim_{n \rightarrow \infty} K_c(\varphi_n, D_1 \setminus \{p_1, p_2, \dots, p_m\}) &\geq K_c(\varphi_0, D_1 \setminus \{p_1, p_2, \dots, p_m\}) + \sum_{s=1}^m K_{p_s}.\end{aligned}$$

Let D_1 and D_2 approach D , and use a similar diagonalize argument, we can show that the theorem holds true. QED.

2.4 Limit of a weak convergence sequence

Proof of theorem 2. We will prove 2.1, 2.2 and 2.3 separately.

(2.1). Let $u = -\ln r = -\ln \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} \frac{y}{x}$. The domain $D \setminus \{0\}$ becomes an infinite cylinder $\{(u, \theta) | 0 \leq u \leq \infty, -\pi \leq \theta \leq \pi\}$ via this transformation. Let $\psi(u, \theta) = \varphi(e^{-u} \cos \theta, e^{-u} \sin \theta) - u$. Then ψ satisfies the following inequalities:

$$\begin{cases} \int_0^\infty \int_{-\pi}^\pi \frac{(\Delta_{u,\theta}\psi)^2}{e^{2\psi}} d\theta du \leq C_2, \\ \int_0^\infty \int_{-\pi}^\pi e^{2\psi} d\theta du \leq C_1, \end{cases} \quad (20)$$

where $\Delta_{u,\theta} = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial \theta^2}$. To prove theorem 2.1, we only need to show that $\psi \rightarrow -\infty$ as $u \rightarrow \infty$. If this is not true, there then exists a positive number C and a sequence of points $\{(u_i, \theta_i), i \in \mathbf{N}\}$ ($u_i \rightarrow \infty$) such that: $\psi(u_i, \theta_i) > -C$. Consider the open disk $\tilde{D} = \{(u, \theta) | -1 < u < 1, -\frac{\pi}{2} < \theta < \frac{\pi}{2}\}$. Define a new sequence of metrics in \tilde{D} as:

$$\varphi_i(u, \theta) = \psi(u + u_i, \theta + \theta_i), \quad \forall i \in \mathbf{N}, \forall (u, \theta) \in \tilde{D}.$$

Then $\{\varphi_i(u, \theta), i \in \mathbf{N}\}$ is a sequence of functions in \tilde{D} with finite energy and area. According to theorem 1, there exists a subsequence $\{\varphi_{n_j}, j \in \mathbf{N}\}$ of $\{\varphi_n\}$ and a metric $\varphi_0 \in \hat{H}_{loc}^{2,2}(\tilde{D} \setminus \{q_1, q_2, \dots, q_l\})$ for some isolated singular points $\{q_1, q_2, \dots, q_l\}$ such that:

$$\varphi_{n_j} \rightharpoonup \varphi_0 \text{ in } \hat{H}_{loc}^{2,2}(\tilde{D} \setminus \{q_1, q_2, \dots, q_l\}), \quad l \geq 0.$$

The vanishing case ($\varphi_0 \equiv -\infty$) does not occur because of

$$\varphi_{n_j}(0, 0) = \psi(u_{n_j}, \theta_{n_j}) > -C, \quad \forall j \in \mathbf{N}.$$

If there exists at least one bubble point $p \in \tilde{D}$, we have:

$$\int_{\tilde{D}} \frac{(\Delta_{u,\theta}\varphi_{n_j})^2}{e^{2\varphi_{n_j}}} du d\theta \cdot \int_{\tilde{D}} e^{2\varphi_{n_j}} du d\theta > \frac{1}{2} E_p \cdot A_p \geq 2\pi^2. \quad (21)$$

If there exists no bubble point, then $\varphi_0 \in H^{2,2}(\tilde{D})$ and $\varphi_{n_j} \rightharpoonup \varphi_0$ in $H^{2,2}(\tilde{D})$. If n is large enough, then:

$$\int_{\tilde{D}} \frac{(\Delta_{u,\theta}\varphi_{n_j})^2}{e^{2\varphi_{n_j}}} d u d \theta \cdot \int_{\tilde{D}} e^{2\varphi_{n_j}} d u d \theta > \frac{1}{2} \int_{\tilde{D}} \frac{(\Delta_{u,\theta}\varphi_0)^2}{e^{2\varphi_0}} d u d \theta \cdot \int_{\tilde{D}} e^{2\varphi_0} d u d \theta > 0. \quad (22)$$

However,

$$\begin{aligned} \int_{\tilde{D}} \frac{(\Delta_{u,\theta}\varphi_{n_j})^2}{e^{2\varphi_{n_j}}} d u d \theta \cdot \int_{\tilde{D}} e^{2\varphi_{n_j}} d u d \theta &= \int_{u_{n_j}-1}^{u_{n_j}+1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(\Delta_{u,\theta}\psi)^2}{e^{2\psi}} d u d \theta \cdot \int_{u_{n_j}-1}^{u_{n_j}+1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2\psi} d u d \theta \\ &\rightarrow 0, \quad \text{as } j \rightarrow \infty. \end{aligned}$$

The last formula holds true because of inequality (20). This contradicts both inequalities (21) and (22). The first part of the theorem is then proved.

(2.2). If $\lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} \varphi'_r(r \cos \theta, r \sin \theta) r d \theta$ does not exist, there then exist two numbers $\alpha \neq \alpha'$ and two alternative sequence of numbers $\{\delta_i\}, \{\delta'_i\}$ such that:

$$\delta_i < \delta'_i < \delta_{i-1} \rightarrow 0, \quad \forall i \in \mathbf{N},$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi'_r(\delta_i \cos \theta, \delta_i \sin \theta) \delta_i d \theta \rightarrow \alpha, \quad \frac{1}{2\pi} \int_0^{2\pi} \varphi'_r(\delta'_i \cos \theta, \delta'_i \sin \theta) \delta'_i d \theta \rightarrow \alpha'.$$

Clearly,

$$A_c(\delta_i, \delta'_i) = \int_{\delta_i}^{\delta'_i} \int_0^{2\pi} e^{2\varphi} r d \theta d r \rightarrow 0.$$

However,

$$\begin{aligned} \left| \int_{\delta_i}^{\delta'_i} \int_0^{2\pi} \Delta \varphi \cdot r \cdot d \theta d r \right| &= \left| \int_{\delta_i}^{\delta'_i} \int_0^{2\pi} (\varphi'_r \cdot r)'_r d \theta d r \right| \\ &= \left| \int_0^{2\pi} \varphi'_r(\delta_i \cos \theta, \delta_i \sin \theta) \delta_i d \theta - \int_0^{2\pi} \varphi'_r(\delta'_i \cos \theta, \delta'_i \sin \theta) \delta'_i d \theta \right| \\ &\rightarrow |\alpha - \alpha'| > 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} C_2 \geq K_c(\delta_i, \delta'_i) &= \frac{\int_{\delta_i}^{\delta'_i} \int_0^{2\pi} \frac{(\Delta \varphi)^2}{e^{2\varphi}} r d \theta d r}{\left| \int_{\delta_i}^{\delta'_i} \int_0^{2\pi} \Delta \varphi \cdot r \cdot d \theta d r \right|^2} \rightarrow \frac{|\alpha - \alpha'|}{0} = \infty. \end{aligned}$$

This is a contradiction. Therefore, $\lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} \varphi'_r(r \cos \theta, r \sin \theta) r d \theta$ does exist.

(2.3). For any small $r = e^{-u} > 0$, consider the domain $\tilde{D} = [-1, 1] \times S^1$. Let $\tilde{\varphi}(v, \theta) = \psi(v + u, \theta)$ (following the notations in (2.1)), then

$$-\Delta_{v,\theta} \tilde{\varphi} = K(v + u, \theta) \cdot e^{2\tilde{\varphi}}, \quad \forall (v, \theta) \in \tilde{D}.$$

There then exists a constant C such that: $\tilde{\varphi} \leq C$. The right hand side is bounded in $L^2(\tilde{D})$. Let w be the solution of

$$\begin{cases} -\Delta w &= K(v+u, \theta)e^{2\tilde{\varphi}}, \\ w|_{\partial D} &= 0. \end{cases}$$

Thus, $\|w\|_{L^\infty}$ is uniformly bounded from above (the bound is actually independent of u , since L^2 norm of $\Delta_{v,\theta} \tilde{\varphi}$ in \tilde{D} uniformly converge to 0 as $u \rightarrow \infty$). The harmonic function $h = \tilde{\varphi} - w$ is bounded from below by a constant $-C$. This follows that there exists a constant $\beta \in (0, 1)$ (independent of u) such that:

$$\sup_{\theta} (C - h(0, \theta)) \leq \frac{1}{\beta} \inf_{\theta} (C - h(0, \theta)),$$

Or,

$$\sup_{\theta} \tilde{\varphi}(0, \theta) \leq \frac{1}{\beta} \inf_{\theta} \tilde{\varphi}(0, \theta) + C.$$

In other words,

$$\sup_{\theta} (\tilde{\varphi}(r \cos \theta, r \sin \theta) + \ln r) \leq \frac{1}{\beta} \inf_{\theta} (\tilde{\varphi}(r \cos \theta, r \sin \theta) + \ln r) + C.$$

Integrating on both sides over θ , we obtain:

$$\frac{1}{\beta} (\phi(r) + \ln r) + C_3 \leq \varphi(r \cos \theta, r \sin \theta) + \ln r \leq \beta (\phi(r) + \ln r) + C_4,$$

where C_3, C_4 are two constants independent of r . QED.

2.5 Bubbles on bubbles

Lemma 7 *Suppose D is a coordinate disk with radius $\rho > 0$ and assume $g = e^{2\varphi}(dx^2 + dy^2)$ is a metric on D with finite energy C_2 . For any $\epsilon > 0$, there then exists a constant $C_\epsilon > 0$, which depends only on D, ϵ such that if*

$\max_{r \leq \rho} \int_0^{2\pi} e^{\varphi(r \cos \theta, r \sin \theta)} r d\theta < \epsilon$, then the following holds true:

$$\int_D e^{2\varphi} dx dy \leq C_\epsilon, \quad \text{and} \quad \lim_{\epsilon \rightarrow 0} C_\epsilon = 0.$$

Proof. If the lemma is false, then there exists a sequence of metrics $\{\varphi_n, n \in \mathbf{N}\}$ such that:

$$\begin{cases} \int_D \frac{(\Delta \varphi_n)^2}{e^{2\varphi_n}} dx dy < C, \\ \int_D e^{2\varphi_n} dx dy = 1, \\ \max_{r \leq \rho} \int_0^{2\pi} e^{\varphi_n(r \cos \theta, r \sin \theta)} r d\theta = \epsilon_n \rightarrow 0. \end{cases} \quad (23)$$

Any circle ($|z| = \delta > 0$) must have a zero length in the limit. Therefore, φ_n vanishes identically except at the origin ($z = 0$). All of the area concentrates at the origin since the total area is fixed. Let $\varepsilon > 0$ be a very small positive number and let $\{\delta_n\}$ be a sequence of numbers such that:

$$\int_{r \leq \delta_n} e^{2\varphi_n} dx dy = \varepsilon < \frac{2\pi^2}{C_2}.$$

Define a new sequence of metrics $F_n = e^{2w_n}(dx^2 + dy^2)$ as:

$$w_n(z) = \varphi_n(\delta_n \cdot z) + \ln \delta_n, \quad \forall r \leq 1.$$

For this new sequence of metrics, we have

$$\begin{aligned} \int_{D_1(0)} e^{2w_n} &= \int_{r \leq \delta_n} e^{2\varphi_n} = \varepsilon, \\ \int_{D_1(0)} \frac{(\Delta w_n)^2}{e^{2w_n}} dx dy &= \int_{r \leq \delta_n} \frac{(\Delta \varphi_n)^2}{e^{2\varphi_n}} dx dy \leq C_2. \end{aligned}$$

By theorem 1, there exists a subsequence which locally weak converges to a metric except at a set of finite number of bubble points. Since each circle $|z| = \delta > 0$ has length 0, the only bubble point must be the original point $z = 0$ and all the area concentrated at 0 must be less than ε . Lemma 2 implies that total energy concentration at the origin must be bigger than $\frac{4\pi^2}{\varepsilon} \geq 2 \cdot C_2$, which is a contradiction. This lemma is then proved. QED.

Corollary 1 *Suppose $\{\varphi_n\}$ is a sequence of metrics with finite area C_1 and energy C_2 . There exists a constant ϵ_0 such that if*

$$\max_{r \leq \rho} |\partial D_r|_{g_n} = \max_{r \leq \rho} \int_0^\pi e^{\varphi_n(r \cos \theta, r \sin \theta)} \cdot r d\theta \leq \epsilon_0, \quad \forall n \in \mathbf{N},$$

then the sequence of metrics does not have any bubble points in D .

Proof: Let C_ϵ be the constant defined according to the previous lemma. We may choose ϵ_0 so small that C_{ϵ_0} satisfies:

$$C_\epsilon \cdot C_2 \leq 2\pi^2.$$

According to the previous lemma, we have:

$$A_c(\varphi_n, D) \leq C_{\epsilon_0} \leq \frac{2\pi^2}{C_2}.$$

Thus the number of possible points m must be bounded by

$$m \leq \sqrt{\frac{C_{\epsilon_0} \cdot C_2}{4\pi^2}} < 1.$$

Therefore, this sequence of metrics has no bubble points. The corollary is then proved. QED.

Proof of theorem 3. Choose any small positive number $\epsilon_0 \in (0, \epsilon)$. This number ϵ serves as a scaling constant (filter). The sequence of functions can be modified slightly so that the following holds true:

$$\varphi_n(p) = \max_{q \in D_{r_0}} \varphi_n(q). \quad (24)$$

Following theorem 2, in a non-vanishing case, we have $\lim_{r \rightarrow 0} \max_{0 \leq \theta \leq 2\pi} (\varphi_0(r \cos \theta, r \sin \theta) + \ln r) = -\infty$. There then exists a number $r_1 > 0$ such that:

$$\max_{0 \leq \theta \leq 2\pi} (\varphi_0(r \cos \theta, r \sin \theta) + \ln r) \ll \epsilon, \quad \forall r < r_1.$$

If n is large enough,

$$\max_{0 \leq \theta \leq 2\pi} (\varphi_n(r_1 \cos \theta, r_1 \sin \theta) + \ln r_1) \ll \epsilon, \quad \forall n > N, \quad (25)$$

or the length of the circle $|z| = r_1$ is very small:

$$|\partial D_{r_1}|_{g_n} = \int_0^{2\pi} e^{\varphi_n(r_1 \cos \theta, r_1 \sin \theta)} r_1 d\theta \ll \epsilon, \quad n > \mathbf{N}. \quad (26)$$

According to corollary 1, if ϵ is small enough, we can choose δ_n such that:

$$|\partial D_r|_{g_n} = \int_0^{2\pi} e^{\varphi_n(r \cos \theta, r \sin \theta)} r d\theta < \epsilon, \quad \forall r_1 \geq r \geq \delta_n, \quad (27)$$

and

$$|\partial D_{\delta_n}|_{g_n} = \int_0^{2\pi} e^{\varphi_n(\delta_n \cos \theta, \delta_n \sin \theta)} \delta_n d\theta = \epsilon. \quad (28)$$

Re-normalize this sequence of metrics as:

$$\phi_n(z) = \varphi_n(\delta_n \cdot z) + \ln \delta_n, \quad \forall |z| < \frac{1}{\delta_n}. \quad (29)$$

For any $n > 0$, ϕ_n is then defined in the disk $D_{\delta_n^{-1}}(0)$. For any fixed number $r > 0$, ϕ_n is well defined on $D_r(0)$ if n is large enough. Moreover, $\{\phi_n\}$ has a finite amount of energy and area since

$$\begin{cases} \int_{D_{\delta_n^{-1}}} \frac{(\Delta \phi_n)^2}{e^{2\phi_n}} dx dy &= \int_{D_1} \frac{(\Delta \varphi_n)^2}{e^{2\varphi_n}} dx dy \leq C_2, \\ \int_{D_{\delta_n^{-1}}} e^{2\phi_n} dx dy &= \int_{D_1} e^{2\varphi_n} dx dy \leq C_1. \end{cases} \quad (30)$$

Applying theorem 1 successively to $\{\phi_n\}$ in a sequence of disks $D_{2^j}(0)$ ($j = 1, 2, \dots$). In disk $D_2(0)$, there exists a subsequence of $\{\phi_{1n}, n \in \mathbf{N}\}$ of $\{\phi_n, n \in \mathbf{N}\}$,

a finite number of bubble points $s_1 = \{p_{11}, p_{12}, \dots, p_{1m_1}\} (m_1 \geq 0)$ with respect to this subsequence, and a metric $\phi_{0,1} \in \hat{H}_{loc}^{2,2}(D_2(0) \setminus \{p_{11}, p_{12}, \dots, p_{1m_1}\})$ such that:

$$\phi_{1n} \rightharpoonup \phi_{0,1} \text{ in } \hat{H}_{loc}^{2,2}(D_2(0) \setminus \{p_{11}, p_{12}, \dots, p_{1m_1}\}),$$

Consider the sequence $\{\varphi_{1n}\}$ in disk $D_{2^2}(0)$. There exists a subsequence $\{\phi_{2n}\}$ of $\{\phi_{1n}\}$, a finite number of bubble points $s_2 = \{p_{21}, p_{22}, \dots, p_{2m_2}\} (m_2 \geq 0)$ with respect to this subsequence, and a metric $\phi_{0,2} \in \hat{H}^{2,2}(D_{2^2}(0) \setminus \{p_{21}, p_{22}, \dots, p_{2m_2}\})$ such that:

$$\phi_{2n} \rightharpoonup \phi_{0,2} \text{ in } \hat{H}_{loc}^{2,2}(D_{2^2}(0) \setminus \{p_{21}, p_{22}, \dots, p_{2m_2}\}),$$

Clearly, the set $s_1 = \{p_{11}, p_{12}, \dots, p_{1m_1}\}$ is a subset of $s_2 = \{p_{21}, p_{22}, \dots, p_{2m_2}\}$ and $m_2 \geq m_1$. Moreover, $\phi_{0,2} = \phi_{0,1}$ when both functions are restricted to the smaller domain D_2 . In particular, $\phi_{0,1} \equiv -\infty$ if and only if $\phi_{0,2} \equiv -\infty$.

In general, suppose that for any $i \leq j$, a subsequence $\{\phi_{in}\}$ had been selected, and a limit metric $\{\phi_{0,i}, i \leq j\}$ had been defined in $\hat{H}^{2,2}(D_{2^i}(0) \setminus s_i)$ where s_i is the set of bubble points of $\{\phi_{in}\}$ in $D_{2^i}(0)$. Consider the subsequence $\{\phi_{jn}\}$ in $D_{2^{j+1}}(0)$. There exists a subsequence $\{\varphi_{(j+1)n}\}$ of $\{\varphi_{jn}\}$, a finite number of bubble points $s_{j+1} = \{p_{(j+1)1}, p_{(j+1)2}, \dots, p_{(j+1)m_{j+1}}\}$ with respect to this subsequence, and a limit metric $\phi_{0,j+1} \in \hat{H}^{2,2}(D_{2^{j+1}}(0) \setminus \{p_{(j+1)1}, p_{(j+1)2}, \dots, p_{(j+1)m_{j+1}}\})$ such that:

$$\phi_{(j+1)n} \rightharpoonup \phi_{0,j+1} \text{ in } \hat{H}_{loc}^{2,2}(D_{2^{j+1}}(0) \setminus \{p_{(j+1)1}, p_{(j+1)2}, \dots, p_{(j+1)m_{j+1}}\}). \quad (31)$$

Consider the diagonal subsequence $\{\phi_{nn}\}$. This is a subsequence of all the previous subsequences $\{\phi_{jn}, n \in \mathbf{N}\}$ for $j = 1, 2, \dots$. Therefore, all of the previous weak convergent results hold true for this subsequence. In particular, the following three statements (for any $j > i \geq 1$) hold true:

1. $s_i \subset s_j$.
2. $\phi_{0,i} \equiv -\infty$ if and only if $\phi_{0,j} \equiv -\infty$.
3. $\phi_{0,j}|_{D_{2^i}} = \phi_{0,i}$ if neither of two metrics vanishes.

Following proposition 2, $m_j \leq \sqrt{\frac{C_1 \cdot C_2}{4\pi^2}}(\forall j)$. There then exists a number N such that $s_j = s_N, \forall j > N$. We may assume that set of bubble points is: $s_N = \{q_1, q_2, \dots, q_m\} = \bigcup_j s_j (m = m_N \geq 0)$.

Define a function ϕ_0 in $S^2 \setminus \{\infty, q_1, q_2, \dots, q_m\}$ by:

$$\phi_0(p) = \phi_{0,j}(p), \quad \forall p \in D_{2^j}(0) \setminus \{q_1, q_2, \dots, q_m\}.$$

Thus, $\phi_0 \in \hat{H}_{loc}^{2,2}(S^2 \setminus \{\infty, p_1, p_2, \dots, p_m\})$ and the following statement holds true:

$$\phi_{nn} \rightharpoonup \phi_0 \text{ in } \hat{H}_{loc}^{2,2}(S^2 \setminus \{\infty, q_1, q_2, \dots, q_m\}).$$

For simplicity, we re-label $\{\phi_{nn}\}$ as $\{\phi_n\}$. Let r be any number large enough so that $\{q_1, q_2, \dots, q_m\} \subset D_r(0)$. Consider the sequence of functions $\{\phi_n\}$ in $D_r(0)$. Suppose the concentrations of area and energy in the bubble point q_i are A_i and K_i . According to theorem 1, we have:

$$\lim_{n \rightarrow \infty} \int_{D_r(0)} e^{2\phi_n} dx dy = \int_{D_r(0)} e^{2\phi_0} dx dy + \sum_{i=1}^m A_i. \quad (32)$$

On the other hand,

$$\int_{D_r} e^{2\phi_n} dx dy = \int_{D_{\delta_n \cdot r}} e^{2\phi_n} dx dy \quad (33)$$

Choose a sequence of numbers $\{\epsilon_i \searrow 0, i \in \mathbf{N}\}$. According to the proof of proposition 1, we may have (passing to a subsequence if necessary):

$$A_p(\epsilon_i) = \lim_{n \rightarrow \infty} A_c(\phi_n, D_{\epsilon_i}(p)), \quad K_p(\epsilon_i) = \lim_{n \rightarrow \infty} K(\phi_n, D_{\epsilon_i}(p)), \quad \forall i \in \mathbf{N},$$

and

$$A_p = \lim_{n \rightarrow \infty} A_c(\epsilon_i), \quad K_p = \lim_{n \rightarrow \infty} K(\epsilon_i).$$

For any fixed i , then $\delta_n \cdot r < \epsilon_i$ if n is large enough. Equation (33) then implies:

$$A_c(\phi_n, D_r(p)) = A_c(\phi_n, D_{\delta_n \cdot r}(p)) \leq A_c(\phi_n, D_{\epsilon_i}(p)).$$

Taking the limit on both sides as $n \rightarrow \infty$, the result is:

$$\lim_{n \rightarrow \infty} A_c(\phi_n, D_r) \leq A_p(\epsilon_i), \quad \forall i \in \mathbf{N}.$$

Taking limit on both sides as $i \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} A_c(\phi_n, D_r) \leq A_p.$$

Similarly,

$$\lim_{n \rightarrow \infty} K_c(\phi_n, D_r) \leq K_p.$$

This implies that $m \leq \sqrt{\frac{K_p \cdot A_p}{4\pi^2}}$. Applying Theorem 1 for $\{\phi_n\}$ in D_r , we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} A_c(\phi_n, D_r) &= A_c(\phi_0, D_r \setminus \{q_1, q_2, \dots, q_m\}) + \sum_{i=1}^m A_{q_i} \\ \lim_{n \rightarrow \infty} K(\phi_n, D_r) &\geq K_c(\phi_0, D_r \setminus \{q_1, q_2, \dots, q_m\}) + \sum_{i=1}^m K_{q_i}. \end{aligned}$$

Thus,

$$\begin{aligned} A_p &\geq A_c(\phi_0, D_r \setminus \{q_1, q_2, \dots, q_m\}) + \sum_{i=1}^m A_{q_i} \\ K_p &\geq K(\phi_0, D_r \setminus \{q_1, q_2, \dots, q_m\}) + \sum_{i=1}^m K_{q_i}. \end{aligned}$$

Let $r \rightarrow \infty$, then:

$$\begin{aligned} A_p &\geq A_c(\phi_0, S^2 \setminus \{\infty, q_1, q_2, \dots, q_m\}) + \sum_{i=1}^m A_{q_i} \\ K_p &\geq K(\phi_0, S^2 \setminus \{\infty, q_1, q_2, \dots, q_m\}) + \sum_{i=1}^m K_{q_i}. \end{aligned}$$

If a vanishing case occurs in $D_{2^j}(0)$ for some j , then it occurs in any disk $D_{2^i}(0)$. Observe the following inequalities:

$$\begin{aligned} 2\pi \cdot e^{\phi_n(0)} &\geq \int_0^{2\pi} e^{\phi_n(\cos \theta, \sin \theta)} d\theta \\ &= \int_0^{2\pi} e^{\varphi_n(\delta_n \cos \theta, \delta_n \sin \theta)} \delta_n \cdot d\theta = \varepsilon. \end{aligned}$$

The first inequality holds true because of equation (24). The last two equalities holds true because of equation (28) and (29). According to lemma 4 (in a vanishing case), the following two statements hold true: (1) p is a bubble point of $\{\phi_n\}$; (2) there exists at least one bubble point in the unit circle. Thus, in a vanishing case, $m \geq 2$.

QED.

3 Geometrical Consequence

3.1 Theorem of weak convergence

A Riemannian metric is said to be a “limit metric” if it is a weak limit of a sequence of Riemannian metrics in $H^{2,2}(\Omega)$. Lemma 6 implies that a limit metric vanishes at one point if and only if it vanishes everywhere in its domain. For the convenience of notations, we add the “0” metric into $H^{2,2}(\Omega)$ and the resulting space is denoted by $\hat{H}^{2,2}(\Omega)$. Assume the following:

$$K(0, \Omega) = A(0, \Omega) = 0, \quad \text{for any sub-domain } \Omega.$$

A sequence of Riemannian metrics $\{g_n\} \in H^{2,2}(\Omega)$ weakly converges to a limit metric g_0 in $\hat{H}_{loc}^{2,2}(\Omega)$ if and only if one of the following two alternatives holds true (mutually exclusive):

1. (Vanishing case). If $g_0 \equiv 0$, then $g_n \rightarrow 0$ everywhere.
2. (Non-vanishing case). If $g_0 \neq 0$, then $\varphi_n \rightharpoonup \varphi_0$ in $H_{loc}^{2,2}(\Omega)$, where $g_n = e^{2\varphi_n} g_{bk}$, $g_0 = e^{2\varphi_0} g_{bk}$; and g_{bk} is a smooth background metric in Ω .

We are now ready to re-state the theorem 1 in a geometric context:

Theorem 1'. *Let $\{g_n, n \in \mathbf{N}\}$ be a sequence of metrics with a finite area C_1 and energy C_2 in a coordinate disk D . There exists a subsequence $\{g_{n_j}, j \in \mathbf{N}\}$ of $\{g_n\}$, a finite number of bubble points $\{p_1, p_2, \dots, p_m\}$ ($0 \leq m \leq \sqrt{\frac{C_1 \cdot C_2}{4\pi^2}}$) with respect to $\{g_{n_j}, j \in \mathbf{N}\}$, and a limit metric g_0 in $\hat{H}^{2,2}(D \setminus \{p_1, p_2, \dots, p_m\})$ such that:*

$$g_{n_j} \rightharpoonup g_0 \text{ in } \hat{H}_{loc}^{2,2}(D \setminus \{p_1, p_2, \dots, p_m\}).$$

If the amount of area and energy concentrations of $\{g_{n_j}\}$ at each point p_i are A_{p_i} and K_{p_i} , then:

$$\lim_{j \rightarrow \infty} A(g_{n_j}, D) = A(g_0, D \setminus \{p_1, p_2, \dots, p_m\}) + \sum_{i=1}^m A_{p_i} \quad (34)$$

$$\lim_{j \rightarrow \infty} K(g_{n_j}, D) \geq K(g_0, D \setminus \{p_1, p_2, \dots, p_m\}) + \sum_{i=1}^m K_{p_i}. \quad (35)$$

Proof. Re-write the sequence of metrics in a fixed coordinate system as:

$$g_n = e^{2\varphi_n}(dx^2 + dy^2).$$

Thus, $\{\varphi_n, n \in \mathbf{N}\}$ is a sequence of metrics with finite area C_1 and energy C_2 . The rest of the proof is a direct translation of the proof of theorem 1 on p. 10. QED.

Theorem 5 Let $\{g_n, n \in \mathbf{N}\}$ be a sequence of Riemannian metrics in M (M is any open surface) with a finite area C_1 and energy C_2 . There exists a subsequence of $\{g_n\}$, a finite number of bubble points $\{p_1, p_2, \dots, p_m\}$ ($0 \leq m \leq \sqrt{\frac{C_1 \cdot C_2}{4\pi^2}}$) with respect to this subsequence, and a limit metric g_0 such that:

$$g_n \rightharpoonup g_0 \text{ in } \hat{H}_{loc}^{2,2}(M \setminus \{p_1, p_2, \dots, p_m\}).$$

If the amount of area and energy concentrations at each point p_i are A_{p_i} and K_{p_i} , then:

$$\lim_{j \rightarrow \infty} A(g_{n_j}, M) = A(g_0, M \setminus \{p_1, p_2, \dots, p_m\}) + \sum_{i=1}^m A_{p_i} \quad (36)$$

$$\lim_{j \rightarrow \infty} K(g_{n_j}, M) \geq K(g_0, M \setminus \{p_1, p_2, \dots, p_m\}) + \sum_{i=1}^m K_{p_i}. \quad (37)$$

Proof: Let $\{U_1, U_2, \dots, U_n, \dots\}$ be a locally finite covering of M where each U_j is a coordinate disk. Consider the restrictions of the sequence of metrics $\{g_n\}$ in each U_j . These metrics have a finite area C_1 and a finite energy C_2 . Apply theorem 1 successively to metrics in each coordinate disk. In U_1 , there exists a subsequence $\{g_{1n}, n \in \mathbf{N}\}$ of $\{g_n\}$, a finite set of bubble points $S_1 = \{q_{11}, q_{12}, \dots, q_{1m_1}\}$ ($0 \leq m_1 \leq \sqrt{\frac{C_1 \cdot C_2}{4\pi^2}}$) with respect to this subsequence, and a limit metric h_1 in $\hat{H}_{loc}^{2,2}(U_1 \setminus S_1)$ such that:

$$g_{1n} \rightharpoonup h_1 \text{ in } \hat{H}_{loc}^{2,2}(U_1 \setminus S_1).$$

Consider this subsequence $\{g_{1n}\}$ in U_2 . There exists a subsequence $\{g_{2n}, n \in \mathbf{N}\}$ of $\{g_{1n}, n \in \mathbf{N}\}$, a finite set of bubble points $S_2 = \{q_{21}, q_{22}, \dots, q_{2m_2}\}$ ($0 \leq m_2 \leq \sqrt{\frac{C_1 \cdot C_2}{4\pi^2}}$) with respect to this subsequence, and a limit metric h_2 in $\hat{H}_{loc}^{2,2}(U_2 \setminus S_2)$ such that:

$$g_{2n} \rightharpoonup h_2 \text{ in } \hat{H}_{loc}^{2,2}(U_2 \setminus S_2).$$

In general, if $\{g_{jn}\}$ had been defined in each coordinate disk $U_i (i \leq j)$, we can select a subsequence $\{g_{(j+1)n}\}$ of $\{g_{jn}\}$ in U_{j+1} so that there is a finite number of bubble points $S_{j+1} = \{q_{(j+1)1}, q_{(j+1)2}, \dots, q_{(j+1)m_{j+1}}\} (0 \leq m_{j+1} \leq \sqrt{\frac{C_1 \cdot C_2}{4\pi^2}})$ in U_{j+1} with respect to this subsequence, and a limit metric h_{j+1} in $\hat{H}_{loc}^{2,2}(U_{j+1} \setminus S_{j+1})$ such that:

$$g_{(j+1)n} \rightharpoonup h_{j+1} \text{ in } \hat{H}_{loc}^{2,2}(U_{j+1} \setminus S_{j+1}).$$

Consider the diagonal subsequence $\{g_{nn}, n \in \mathbf{N}\}$. In each coordinate disk $U_j (\forall j)$, the following holds true:

$$g_{nn} \rightharpoonup h_j \text{ in } \hat{H}_{loc}^{2,2}(U_j \setminus S_j).$$

This set of limit metrics $\{h_j\}$ then defines a limit metric g_0 in $H_{loc}^{2,2}(M \setminus (\bigcup_j S_j))$ by:

$$g_0(p) = h_j(p), \quad \forall p \in U_j.$$

This metrics is well defined since $h_i \equiv h_j$ on $U_i \cap U_j$ if $U_i \cap U_j \neq \emptyset$. Thus,

$$g_{nn} \rightharpoonup g_0 \text{ in } H_{loc}^{2,2}(M \setminus (\bigcup_j S_j)).$$

The cardinality of the set $\bigcup_j S_j$ must be bounded by $\sqrt{\frac{C_1 \cdot C_2}{4\pi^2}}$ according to the proof of proposition 2.

Re-Label this subsequence as $\{g_n\}$. For any two pair of coordinate disks U_i, U_j where $U_i \cap U_j \neq \emptyset$, we have:

$$\lim_{n \rightarrow \infty} A(g_n, U_j \bigcup U_k) = A(g_0, (U_j \bigcup U_k) \setminus (S_j \bigcup S_k)) + \sum_{p \in S_j \bigcup S_k} A_p, \quad (38)$$

$$\lim_{n \rightarrow \infty} K(g_n, U_j \bigcup U_k) \geq K(g_0, (U_j \bigcup U_k) \setminus (S_j \bigcup S_k)) + \sum_{p \in S_j \bigcup S_k} K_p. \quad (39)$$

These two formulas can be readily generalized to any number of coordinate disks:

$$\lim_{n \rightarrow \infty} A(g_n, \bigcup_k U_k) = A(g_0, \bigcup_k U_k \setminus (\bigcup_k S_k)) + \sum_{p \in \bigcup_k S_k} A_p, \quad (40)$$

$$\lim_{n \rightarrow \infty} K(g_n, \bigcup_k U_k) \geq K(g_0, \bigcup_k U_k \setminus (\bigcup_k S_k)) + \sum_{p \in \bigcup_k S_k} K_p. \quad (41)$$

Observed that $M = \bigcup_k U_s$ and $\bigcup_k S_k = \{p_1, p_2, \dots, p_m\}$. QED.

3.2 Blowing up procedure and tenuously connected sum

Let us re-state theorem 3 in the geometric context.

Theorem 3' (Bubbles on bubbles). *Let $\{g_n, n \in \mathbf{N}\}$ be a sequence of metrics in D with a finite area C_1 and energy C_2 . Suppose that $p = 0$ is the only bubble point in D with area concentration A_p and energy concentration K_p . Fix a local z -coordinate system centered at p and a scaling constant ε . If ε is small enough, we can re-normalize the sequence of metrics by $\tilde{g}_n(z) = g_n(\varepsilon_n \cdot z + z(p_n))$ where $\{\varepsilon_n \searrow 0\}$ is uniquely determined by the scaling constant ε ; where $p_n \rightarrow p$ is the supremum of mass g_n in D . There then exists a subsequence of $\{g_n\}$, a finite number of bubble points $\{q_1, q_2, \dots, q_m\}$ ($0 \leq m \leq \sqrt{\frac{A_p K_p}{4\pi^2}}$) in $S^2 \setminus \{\infty\}$ with respect to the corresponding subsequence of $\{\tilde{g}_n\}$, and a limit metric \tilde{g}_0 in $\hat{H}^{2,2}(S^2 \setminus \{\infty, q_1, q_2, \dots, q_m\})$ such that:*

$$\tilde{g}_n \rightarrow \tilde{g}_0 \text{ in } \hat{H}_{loc}^{2,2}(S^2 \setminus \{\infty, q_1, q_2, \dots, q_m\}).$$

If the amount of area and energy concentrations of $\{\tilde{g}_n\}$ at each q_i are A_{q_i} and K_{q_i} respectively, then:

$$A_p \geq A(\tilde{g}_0, S^2 \setminus \{\infty, q_1, q_2, \dots, q_m\}) + \sum_{i=1}^m A_{q_i}, \quad (42)$$

$$K_p \geq K(\tilde{g}_0, S^2 \setminus \{\infty, q_1, q_2, \dots, q_m\}) + \sum_{i=1}^m K_{q_i}. \quad (43)$$

Let us review the steps taken in the proof of theorem 3. For convenience, we use a complex notation. The metric can be expressed as:

$$g_n(z) = e^{2\varphi_n(z)} |dz|^2, \quad \forall n \in \mathbf{N}.$$

The first step is to move the supremum of mass of the metric g_n to the center of the coordinate system. If $\{p_n\}$ is such a sequence of points, then define

$$\tilde{g}_n(z) = g_n(z + z(p_n)), \quad z \in D_1(p).$$

Re-Label $\{\tilde{g}_n\}$ as $\{g_n\}$. The supremum of the metric g_n is now at p , $\forall n \in \mathbf{N}$.

Choose a small positive number $\varepsilon < \epsilon_0$ (as in corollary 1) as a filter. Following the proof of theorem 3, there then exists a number $r_1 > 0$ such that if n is large enough, we have:

$$\max_{0 \leq \theta \leq 2\pi} (\varphi_n(r_1 \cos \theta, r_1 \sin \theta) + \ln r_1) \ll \varepsilon, \quad \forall n > N, \quad (44)$$

or the length of this circle at $|z| = r_1$ is very small:

$$\int_0^{2\pi} e^{\varphi_n(r_1 \cdot z)} r_1 d\theta \ll \varepsilon, \quad n > \mathbf{N}. \quad (45)$$

Following corollary 1, there exists $\delta_n > 0$ such that:

$$\int_0^{2\pi} e^{\varphi_n(r \cdot z)} r d\theta < \varepsilon, \quad \forall r_1 \geq r \geq \delta_n, \quad (46)$$

and

$$\int_0^{2\pi} e^{\varphi_n(\delta_n \cdot z)} \delta_n d\theta = \varepsilon. \quad (47)$$

The circle $|z| = \delta_n$ is the first circle for which the metric g_n has a length of ε beyond a thin neck. Hence, the set of concentric circles $\{|z| = \delta_n\}$ is uniquely determined by the filter size ε once the local coordinate system is picked. Define a sequence of conformal parameter functions as:

$$\phi_n(z) = \varphi_n(\delta_n \cdot z) + \ln \delta_n, \quad n \in \mathbf{N}.$$

Thus re-normalize the original sequence of metrics as

$$\tilde{g}_n(z) = e^{2\phi_n(z)} |dz|^2 = g_n(\delta_n \cdot z), \quad n \in \mathbf{N}.$$

Theorem 3 then asserts that we could choose a subsequence $\{\varphi_{n_i}, i \in \mathbf{N}\}$ of $\{\varphi_n, n \in \mathbf{N}\}$, a finite number of bubble points $\{q_1, q_2, \dots, q_m\}$ ($0 \leq m \leq \sqrt{\frac{A_p \cdot K_p}{4\pi^2}}$) such that one of the following two alternatives holds true:

1. $\phi_{n_j}(z) \rightarrow -\infty$ in $S^2 \setminus \{\infty, q_1, q_2, \dots, q_m\}$.
2. There exists a metric $\phi_0 \in H_{loc}^{2,2}(S^2 \setminus \{\infty, q_1, q_2, \dots, q_m\})$ such that

$$\phi_{n_j}(z) \rightharpoonup \phi_0, \text{ in } H_{loc}^{2,2}(S^2 \setminus \{\infty, q_1, q_2, \dots, q_m\}).$$

Define $\tilde{g}_0 \equiv 0$ in a vanishing case; and define $\tilde{g}_0 = e^{2\phi_0} |dz|^2$ in a non-vanishing case. Thus,

$$\tilde{g}_{n_j} \rightharpoonup \tilde{g}_0 \text{ in } \hat{H}_{loc}^{2,2}(S^2 \setminus \{\infty, q_1, q_2, \dots, q_m\}).$$

Moreover,

$$\tau_p = A_p - A(g_0, S^2 \setminus \{\infty, q_1, q_2, \dots, q_m\}) - \sum_{i=1}^m A_{q_i} \geq 0, \quad (48)$$

$$K_p \geq K(g_0, S^2 \setminus \{\infty, q_1, q_2, \dots, q_m\}) + \sum_{i=1}^m K_{q_i}, \quad (49)$$

where τ_p denote the amount of area lost in the neck during the re-normalization (blowing up) process.

Choose r_2 big enough, so that $\{q_1, q_2, \dots, q_m\} \subset D_{r_2}$. Consider the cylinder bounded by the two concentric circles $|z| = r_1$ and $|z| = r_2 \cdot \delta_n$. This cylinder is called the “neck” of the blowing up process. By definition, the length of a circle in this cylinder is bounded above by ε . As $n \rightarrow \infty$, the conformal distance between

the two boundary circles approaches ∞ , while part of the interior of the neck collapses into a line. The collapsing can occur either by keeping the scalar curvature point-wisely bounded, or by keeping the diameter of the neck bounded. Denoted this neck by $Neck(r_1, r_2)$. We can shrink the size of the neck by letting $r_1 \rightarrow 0$ and $r_2 \rightarrow \infty$.

This blowing up procedure or the re-normalization procedure depends only on the filter size $\varepsilon > 0$ once a coordinate system is fixed. Suppose that g_0 is a limit metric in $\hat{H}^{2,2}(D \setminus \{p\})$ such that:

$$g_n \rightharpoonup g_0 \text{ in } \hat{H}_{loc}^{2,2}(D \setminus \{p\}).$$

The surface $(g_0, D \setminus p)$ and $(\tilde{g}_0, S^2 \setminus \{\infty, q_1, q_2, \dots, q_m\})$ are called tenuously connected at p and at $z = \infty$. If $\tau_p = 0$, the connected sum is then efficient. Otherwise the connected sum is inefficient.

The following is an example of a “tenuously connected sum” of two Riemannian metrics or surfaces:

Example 2. *We first construct a metric in a disk, where the boundary curve is a closed geodesic. We can make the length of the boundary approaches 0, while keeping the area and energy finite (see Figure 4 below). The following is a sketch of the construction. Suppose that $g = e^{2\varphi}|dz|^2$ is a rotationally symmetric metric defined in \mathbf{R}^2 (real plane) such that:*

$$\varphi(r) = -\ln r - \beta \cdot \ln(\ln r), \quad \forall r > 2,$$

where $\frac{1}{2} < \beta < \frac{3}{2}$. Let $\epsilon_n = \frac{\beta}{\ln n}$, $\delta_n = \frac{\beta}{\ln^2 n}$, and $T_n = n + \ln n$. Define a sequence of metrics $\{\varphi_n\}$ in D_{T_n} as the following:

$$\varphi_n(r) = \begin{cases} \varphi(r), & \text{when } r \leq n \\ \varphi(n) + \ln n - \ln r - \epsilon_n(r - n) + \frac{1}{2}\delta_n(r - n)^2, & \text{when } n \leq r \leq T_n \end{cases}$$

It is straightforward to prove that $|z| = T_n$ is a closed geodesic of φ_n and

$$\lim_{n \rightarrow \infty} \varphi_n(r) = \varphi(r), \quad \text{if } r \text{ is finite,}$$

and

$$\lim_{n \rightarrow \infty} E_c(\varphi_n, D_{T_n}) = E_c(\varphi, \mathbf{R}^2), \quad \lim_{n \rightarrow \infty} A_c(\varphi_n, D_{T_n}) = A_c(\varphi, \mathbf{R}^2).$$

Gluing two identical copy of φ_n along the curve $|z| = T_n$, we obtain a metric g_n in S^2 . Clearly, $\{g_n\}$ has finite energy and area, and it weakly converges to g everywhere except at near $z = \infty$. If we blow up the sequence near $z = \infty$, we obtain a new sequence of metrics which locally weakly converges to g except at $z = 0$. Relabel this metric as g_∞ . Then the limit tree structure of the weak limit of (S^2, g_n) consists of a root vertex g and a child vertex g_∞ . The corresponding metrics g and

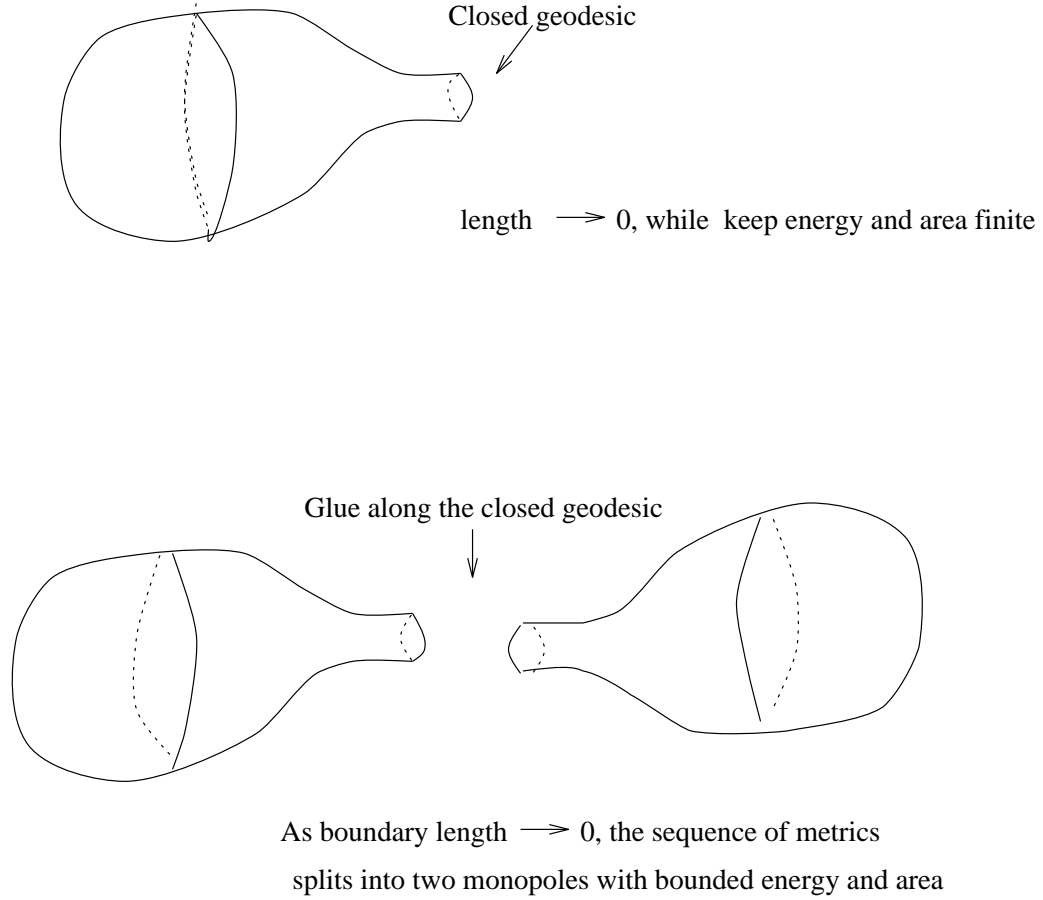


Figure 4: Surface bounded with a small closed geodesic

g_∞ at the two nodes are tenuously connected. This sequence of metrics, clearly has no convergent subsequence in the elementary sense, even up to the Möbious group. Using a similar mechanism, we could construct examples of a sequence of metrics which demonstrates a more sophisticated pattern of limit tree structures.

Lemma 8 *Let g be a metric in $D \setminus \{p\}$ with a finite energy C_2 . Suppose $|\partial D_1|_g = \epsilon$ where $D_1 \subset D \setminus \{p\}$. There exists a constant $C_\epsilon > 0$ (independent of metric g) such that:*

$$A(g, D \setminus \{p\}) = \int_{D \setminus \{p\}} dg > C_\epsilon.$$

Proof. If the lemma is false, there then exists a sequence of metrics $\{g_n\}$ in $D \setminus \{p\}$ such that:

$$K(g_n, D \setminus \{p\}) < C_2, \quad A(g_n, D \setminus \{p\}) \rightarrow 0, \quad |\partial D_1|_{g_n} = \epsilon > 0.$$

According to theorem 5, there exists a subsequence of $\{g_n\}$, a finite bubble points $\{p_1, p_2, \dots, p_m\} (m \geq 0)$, and a limit metric g_0 in $\hat{H}^{2,2}(D \setminus \{p, p_1, p_2, \dots, p_m\})$

such that:

$$g_n \rightharpoonup g_0 \text{ in } \hat{H}_{loc}^{2,2}(D \setminus \{p, p_1, p_2, \dots, p_m\}).$$

Thus, $m = 0$ since the product of area and energy of this subsequence approaches 0. Moreover, $g_0 \equiv 0$ in $D \setminus \{p\}$ since total area approaches 0. This is also impossible since $|\partial D_1|_{g_n} = \epsilon > 0$. The lemma is then proved. QED.

Proposition 3 (continued from theorem 3). \tilde{g}_0 is as defined in theorem 3. If $\tilde{g}_0 \neq 0$ and $m = 1$, then $\int_{S^2 \setminus \{\infty, p\}} d\tilde{g}_0 > C_\epsilon > 0$, where C_ϵ depends only on C_1, C_2 and the scaling constant ϵ .

Proof: If $\tilde{g}_0 \neq 0$ and $m = 1$, then p must be the only bubble point and $|\partial D_1|_{g_0} = \epsilon > 0$. The proposition then follows the previous lemma. QED.

4 Bubble tree

Theorem A. The limit of any locally weakly convergent sequence of metrics $\{g_k, k \in \mathbf{N}\} \in \mathcal{S}(g_0, C_1, C_2, \Omega)$, consists of the following 4 objects: (1) A finite, rooted tree T , possibly reduced to just the base vertex f . (2) The base vertex $f \in T$ is a limit metric in Ω with a finite number of bubble points $\{p_i\}$ deleted; the edges emanating from the base vertex is $\{p_i\}$; there are three masses associated with each edge: the area concentration a_i , energy concentration e_i and area loss during the blowing up process τ_i ($a_i \cdot e_i \geq 4\pi^2$). (3) Any other vertex f_s is a limit metric defined on $S^2 \setminus \{\infty, p_{si}\}$; the edges emanating from this vertexes are $\{p_{si}\}$; and there are three masses associated with each edge: the area concentration a_{si} , energy concentration e_{si} and area lost during the blowing up process τ_{si} . (4) For each pair of vertexes f_{s_1} and f_{s_2} bounding an common edge in T , they are tenuously connected at the pair of respective singular points. The connected sum is efficient if the area loss associated with that edge is 0. If the tree T consists of only the base vertex f , the sequence of metrics $\{g_k\}$ is then said to have a weak convergent limit in the elementary sense (up to the Möbious group). The number of vertexes whose valence $\neq 2$, is bounded from above ($\leq \sqrt{C_1 \cdot C_2}$). The depth of the tree is also finite in a reasonable sense.

Proof of Theorem A. The tree structure is constructed from a sequence of metrics $\{g_n\}$ in Ω as follows (see Figure 1 on p. 3): First, choose a scaling constant ϵ_0 as a filter for re-normalization process. The $\{g_n\}$ locally weakly converges to f_0 on $\Omega \setminus \{p_1, p_2, \dots, p_m\}$ except a finite number of bubble points $\{p_1, p_2, \dots, p_m\}$. The base vertex of the tree is the metric f_0 , which we re-labeled as f , and the edges emanating from the base vertex are the points $\{p_i\}$. Each edge has an energy mass e_i and area mass a_i , which are the energy and area concentrations at the bubble point p_i . For each p_i , the re-normalization process gives a new sequence of metrics $\{\tilde{g}_n\}$ which locally weakly converge to a metric f_i in $S^2 \setminus \{\infty, p_{ij}, j = 1, 2, \dots, m_i\}$,

with the amount of energy and area concentrated at each point p_{ij} are e_{ij} and a_{ij} . We label each edge as (p_i, e_i, a_i, τ_i) where τ_i represent the amount of area lost at the bubble point p_i during the blowing up process. If $\tau_i = 0$, the blowing up process is then efficient. The edge (p_i, e_i, a_i, τ_i) ($e_i \cdot a_i \geq 4\pi^2$ according to lemma 2) terminates at the vertex f_i which, in turn, is the source of new edges $\{p_{ij}\}$, and so on.

At each vertex $f_I = f_{i_1 \dots i_{k-1} i_k}$ of the tree, use S_I to denote all of the bubble points of this limit metric other than the point $z = \infty$. If f_I is not the base vertex, it must have a parent vertex $f_{I'} = f_{i_1 \dots i_{k-1}}$. The surface $(f_{I'}, S^2 \setminus \{\infty, S_{I'}\})$ (or $(f, M \setminus \{p_1, p_2, \dots, p_m\})$ if $I' = \emptyset$) is tenuously connected to $(f_I, S^2 \setminus \{\infty, S_{I'}\})$. If there is area loss during the blowing up process ($\tau_{I'} \neq 0$), the connected sum is inefficient.

Each vertex f_I has a special property: if it vanishes in any point in its domain, then it vanishes everywhere in its domain. In the case when $f_I \equiv 0$, we call this a ghost vertex. At each ghost vertex other than the base vertex, there exists at least two edges emanating from it. In other other words, the metric has at least two bubble points.

The ghost vertex does appear, as seen in example 3 below. However, there exists at most a finite number of ghost vertexes. Otherwise, consider all the vertexes in the tree that have at least two edges emanating from them. These vertexes must be infinitely many since every ghost vertex has at least two edges emanating from it. There exists an infinite number of edges where no two edges belong to the same branch of the tree. Re-Labeling these edges if necessary, we may assume that these edges are $\{(q_i, e_i, a_i, \tau_i), i \in \mathbf{N}\}$ where $a_i \cdot e_i \geq 4\pi^2$. Therefore,

$$C_1 \geq \sum_i a_i, \quad C_2 \geq \sum_i e_i.$$

Thus,

$$C_1 \cdot C_2 \geq \sum_i a_i \cdot e_i = \sum_i 4\pi^2.$$

The last inequality implies that the number of these vertexes (include all the ghost vertexes) must be finite.

For any other vertex which has only one edge emanating from it, proposition 3 implies that the area of such a vertex is bounded below by a positive constant C_ϵ , which depends only on C_1, C_2 and the scaling constant ϵ . The number of these vertexes is finite as well.

Therefore, the limit tree has only a finite depth. If we reduce the size of the filter, a new vertex might be inserted into the tree structure. However, these new vertexes have only one edge emanating from it. The underlying surface is S^2 with

two opposite points deleted. QED.

Example 3. Let $f = (z-1)(z-2)\cdots(z-m)$ be a holomorphic function. Choose a simply connected domain Ω which contains all zero points of f but no zero points of $f'(z)$. Thus, $g_n(z) = \frac{4 \cdot n^2 \cdot |f'|^2}{(1+n^2 \cdot |f|^2)^2} \cdot |dz|^2$ is a sequence of metrics well defined in Ω with finite area and energy (bounded above by $4\pi \cdot m$). Clearly, g_n weakly converges to 0 everywhere except at $z = 1, 2, \dots, m$. At each point $z = k$, a renormalized sequence of metrics weakly converges to a metric in S^2 with curvature 1. Thus, the bubble tree of (g_n, Ω) consists of 1 ghost base vertex and m first level vertexes, where each first level vertex represents a metric with curvature 1 in S^2 .

Proof of Corollary B. Suppose $\{g_k, k \in \mathbf{N}\}$ is a sequence of metrics with finite area C_1 and energy C_2 . If necessary, we pass to a subsequence so that the weak limit of this sequence has a bubble tree decomposition as described in theorem A. Consider a generic pair of consecutive vertexes (f_I, f_{I_i}) in the bubble tree, where p_i is a bubble point of f_I and the re-normalized sequence of metrics at p_{I_i} weakly converges to f_{I_i} except a few bubble points. Consider the “neck” of this blowing up process. It is a cylinder where the length of each concentric circle is bounded above by the scaling constant ϵ . We call this cylinder a “thin component.” Now iterate thorough each pair of consecutive vertexes, and obtain a finite number of “thin” components (See Figure 5 below). The collection of “thin components” is labeled by I_{thin} . For each fix n , remove all of the “thin components” from M . The resulting surface is a disjoint union of a finite number of connected components. Each connected component is called a “thick component.” Label all of the “thick” components by I_{thick} . Each thick component, together with the restriction of g_n on it, weakly converges to a surface with a finite number of disks deleted. The thick component corresponding to the base vertex is Ω with a few disk deleted. All of the rest of thick components are S^2 with a few disks deleted (When a thick component corresponds to a ghost vertex in the tree decomposition, the limit metric is 0). The size of all deleted disks could be shrinked to 0 by shrinking the size of corresponding blowing up “neck.” QED.

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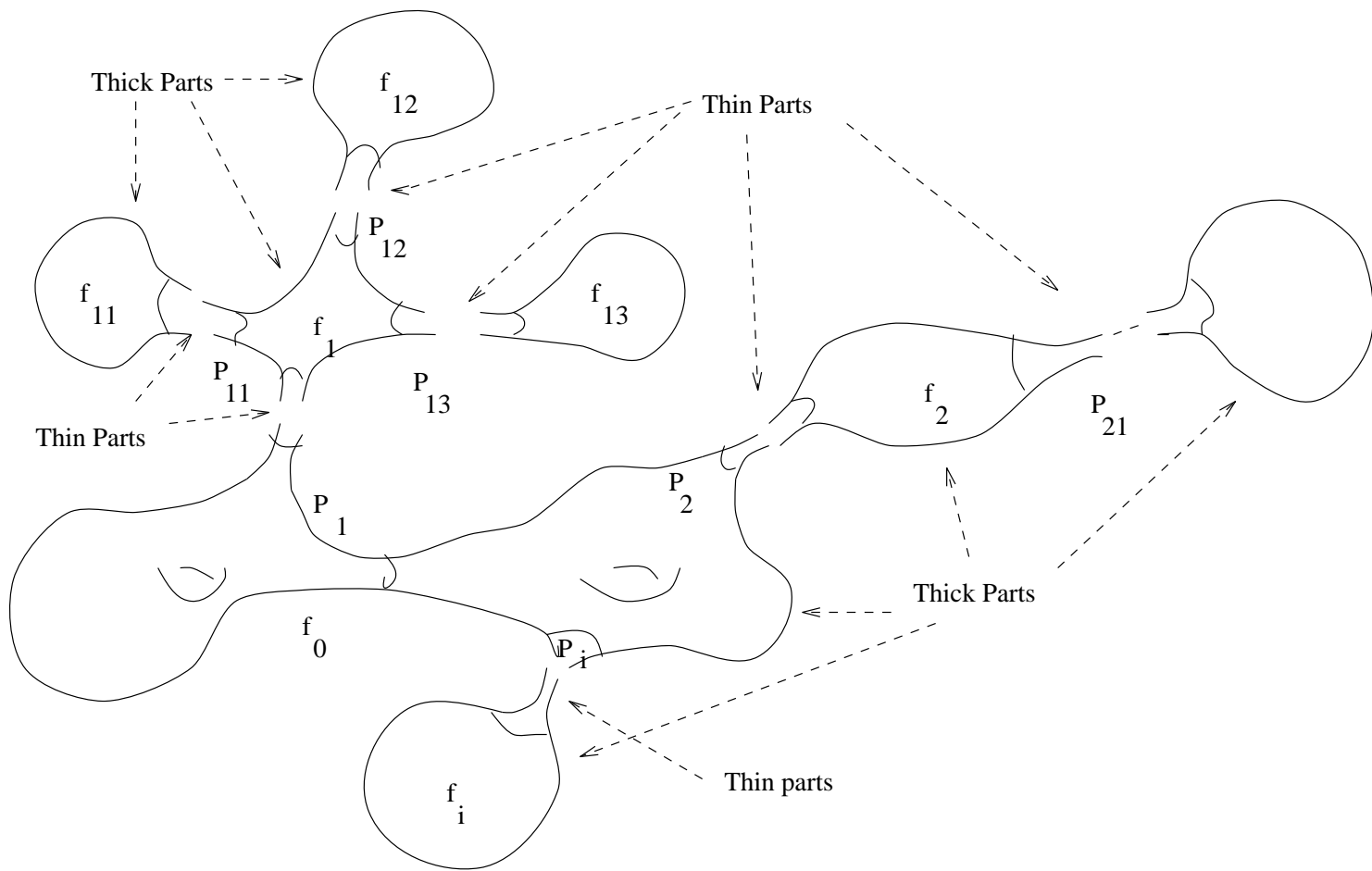


Figure 5: Thin-thick decomposition

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